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BOUNDARY STABILIZATION OF DONNELL'S SHALLOW CIRCULAR CYLINDRICAL SHELL

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Donnell's model of a shallow (and thin) circular cylindrical shell is formulated by a system of three partial differential equations, only one of which contains explicit time dependence. It constitutes one of the most important linear shell models, yet problems associated with its boundary stabilization and control have not been carefully studied. In this paper, we set up the functional–analytic framework, derive dissipative boundary conditions, and determine the infinitesimal generator of the semigroup of evolution. Using a frequency domain method along with energy multipliers, we establish the result of uniform exponential decay of energy under geometric conditions identical to those of the case of a thin Kirchhoff plate. Our approach, incorporating energy multipliers in the frequency domain with a contrapositive argument, appears to be new. It has the beneficial effect of avoiding the necessity to estimate lower order terms when the shell radius is not large. We also consider the case in which the domains contain angular corners; special treatment is required to handle the additional energy contributed by the twisting moments at corner points. Under the assumption of sufficient regularity, uniform exponential decay of energy is also established for such domains.

1. INTRODUCTION

In this paper, we study the boundary stabilization of Donnell's thin circular cylindrical shell.

The basic objects in the study of vibration dynamics in engineering are springs, strings, cables, rods, beams, membranes, plates and shells. The last three among them, namely, membranes, plates and shells, constitute perhaps the most prevalent ones with multi-dimensional space settings. The membrane is modelled by the *second order* wave equation, the analysis and control of which have been rather thoroughly investigated during the past three decades. Next up in the level of mathematical complexity are the various plate models, at least in the sense of the order of partial differential equations involved because they normally (or essentially) have order *four*. Rapid progress on the boundary control, observation and stabilization of plates has been made during the past decade; see the books by Lagnese [1]. Lagnese and Lions [2] and the references therein.

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Now, let us talk about shells. They are as ubiquitous as plate structures in mechanical and civil engineering: aircraft fuselage, ship hulls, pressure vessels, curved roofs and domes, to mention a few. From the modelling point of view, the major difference between plates and shells is that the geometrical effects of *curvature* are significant and have to be taken into account in the mathematical formulation of shells. This results in systems of coupled partial differential equations of even higher order, essentially $\geqslant 8$, cf. reference [11, section 2], for a majority of shallow or thin shells of spherical, conical and cylindrical shapes. For this reason, the mathematical analysis and control of shell models seems to contain almost all of the challenges encountered in the leap from the wave equation to the plate equations.

Even though it has been well recognized that many general techniques in the theory of distributed parameter control are applicable to shell models, so far we are able to locate only a small number of published references on this subject. We may first mention the work by Deng [3], who studied the boundary control of an *axial-symmetric* thin circular cylindrical shell. More recently, we have also noticed the papers by Delfour and Zolesio [4] and Lasiecka, Triggiani and Valente [5]. The primary emphasis in reference [4] is on functional–analytic setting and well-posedness (without any control), while in references [3] and [5], axial/radial symmetries have been used to render the problems to one space dimension. It would certainly be more interesting to treat shell stabilization/control problems in the natural two-dimensional space setting without the axial/radial symmetry assumptions.

In our opinion, the lack of published work presenting systematic mathematical treatments of boundary stabilization and control of shells may be largely attributed to the complexity, and the ensuing somewhat intractable nature, of the shell models involved. Admittedly, the stress–strain theory of shells with various geometries is well developed, as can be read from the large number of shell books in the literature. Thus, it would appear straightforward to derive the PDEs based upon the stress and strain relations (or, equivalently, from the variational principle). This is not exactly so—it is invariably the rule that further *ad hoc* assumptions have to be made to adjust the model by judiciously discarding a few terms because they may not be significant, so that a physically reasonable and acceptable PDE system is obtained that is also "sufficiently mathematically pleasant". Even for the shallow or thin circular cylindrical shell, the reader may find several such *ad hoc* assumptions, as reflected in/associated with the modeller's names of Donnell, Flügge, Timoshenko, Vlasov, Washizu–Goldenveizer, etc. [6]. As a matter of fact, in the authors' struggle to formulate a few such shell PDE systems for treatment, Donnell's shallow circular cylindrical shell model is the only one for which we have achieved success thus far. This model has perhaps the strongest mathematical resemblance to the Kirchhoff thin plate model and therefore, in our opinion, its study constitutes a "natural first leap" from plates to shells. We will provide a brief description of the derivations in section 2.

In section 3, we derive dissipative boundary conditions and provide the functional–analytic setting for the PDE system.

One of the major challenges in studying the shell boundary stabilization problem is the estimation of some lower order terms when the radius R of the circular cylindrical shell is not large, a situation reminiscent of one encountered in Lagnese [7] (and in Chen [8]), where some lower order terms on the boundary need to be absorbed by other negative terms. Here, in section 4, we adopt the frequency domain method along with energy multipliers to prove the uniform exponential decay of energy of Donnell's shallow shell subject to certain geometric conditions on the domain, establishing Main Theorem 1. Although such a frequency domain approach was used in an earlier paper by Chen, Krantz, Ma, Wayne and West [9], based upon an *explicit* representation of the resolvent operator, as well as in a more recent paper by A. Wyler [10], by *direct* estimates using energy multipliers, here our approach is an *indirect contrapositive* argument using multipliers without requiring any explicit knowledge of the resolvent operator, which is more consistent with the argument given in Chen, Fulling, Narcowich and Sun [11]. To our knowledge, this is the first time the frequency domain method is successfully incorporated with the energy multiplier technique and a contrapositive argument to prove exponential stabilization problems. This approach enables us to handle lower order terms (cited at the beginning of this paragraph) with sufficient ease to achieve the desirable result, skipping the kind of unique continuation–compactness arguments that are required in reference [7]. Confer also Liu [12].

In section 5, we study the case of *domain with corners*. When corners are present, the twisting moments there will contribute additional energy terms, and consequently the integration by parts formula needs to be amended (Stern [13], Hartman and Zotemantel [14] and Chen, Coleman and Ding [15]) to take such effects into account. We are able to formulate additional pointwise constraints at corner points, and prove the uniform exponential decay of energy in Main Theorem 2 for domains with corners, under a provisional assumption $[\mathscr{R}]$ of sufficient regularity of solutions.

Since the publication of Bardos, Lebeau and Rauch [16], the method of microlocal analysis has been successfully applied to a variety of time-dependent PDEs to yield uniform decay results on domains with less restrictive geometry (primarily, the kind of geometry that is non-wave trapping). The authors foresee that, perhaps in only a matter of time, such a method will also be successfully applied to the problem under study here. So what is the point of publishing this work, with the understanding that a more advanced method seems destined to surpass it in the (perhaps not remote) future? Our response is articulated as follows:

(1) The microlocal method applies only to domains Ω the boundary $\partial\Omega$ of which is C^∞ (or by taking the limit, C^3), while the energy-multiplier technique (as part of the *a priori* estimates) can be applied to *domains with corners*, when sufficient regularity of solutions is known or assumed, such as the case of rectangular geometry treated in Quinn and Russell [17]. Here our shell model with the presence of corners is even more different from second order PDE systems such as the wave equation, and obviously our Main Theorem 2 is not immediately obtainable from microlocal methods, if our regularity assumption $[\mathscr{R}]$ is indeed valid for some geometries.

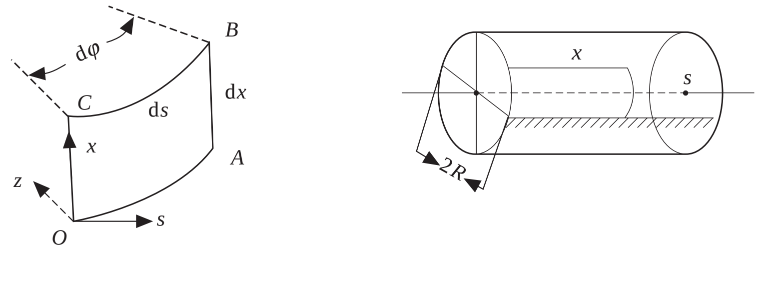
(2) The analysis here still provides useful clues to the future microlocal analysis work.

(3) This classical method is easily usable and understandable, and is accessible to a non-expert in microlocal analysis, whereas the geometrical meanings of assumptions (involving rays) made in microlocal methods are often not as clearly understandable or verifiable.

2. DONNELL'S SHALLOW CIRCULAR CYLINDRICAL SHELL MODEL

We give a brief derivation of Donnell's shallow circular cylindrical model. Consider a differential element $OABC$ as shown in Figure 1(a). Let u be the displacement of the shell in the x-direction of the point O at time t; v, its s-direction displacement; and w, its z-direction displacement. R is the radius of the shell. The co-ordinate system is set up as shown in Figure 1(b).

We denote the membrane forces, shearing forces, bending and twisting moments as shown in Figures 2 and 3.



(a) (b)

Figure 1. (a) The differential element $OABC$. (b) A circular cylindrical shell; $x =$ constant is a circular edge, and $s =$ constant is a straight edge. For a shallow shell to be a good approximation, the overall angle extended from the axis should not be larger than $\pi/3$.

It is known [18–21] that the strains are given by

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_s = \frac{\partial v}{\partial s} + \frac{1}{R} w, \qquad \gamma = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}. \tag{1}$$

Also, the changes in curvature are

$$\kappa_x = \frac{\partial^2 w}{\partial x^2}, \qquad \kappa_s = \frac{1}{R}\frac{\partial v}{\partial s} - \frac{\partial^2 w}{\partial s^2}, \qquad \chi_{xs} = \frac{1}{R}\frac{\partial v}{\partial x} - \frac{\partial^2 w}{\partial x\, \partial s}. \tag{2}$$

Then the stress–strain relations are

$$N_x = \frac{Eh}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_s), \qquad N_s = \frac{Eh}{1-\nu^2}(\varepsilon_s + \nu\varepsilon_x),$$

$$M_x = -D(\kappa_x + \nu\kappa_s), \qquad M_s = -D(\kappa_s + \nu\kappa_x),$$

$$N_{xs} = N_{sx} = \frac{hE}{2(1+\nu)}\gamma, \qquad M_{xs} = -M_{sx} = D(1-\nu)\chi_{xs},$$

where h is the cylinder's thickness, ν its Poisson ratio, $0 < \nu < 1/2$, E the modulus of elasticity, and $D = Eh^3/12(1-\nu)^2$ the flexural rigidity.

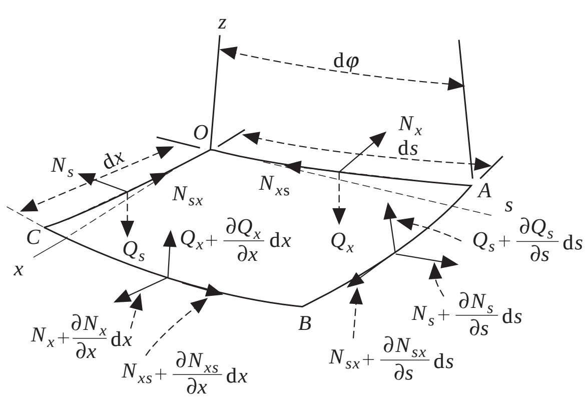


Figure 2. The forces acting on $OABC$. N_x, N_x, N_{xs} (and N_{sx}), membrane forces per unit length of axial section s and x, and a section perpendicular to the axis of a cylindrical shell, respectively; Q_s, Q_x, shearing forces parallel to z-axis per unit length of an axial section and a section perpendicular to the axis of a cylindrical shell, respectively.

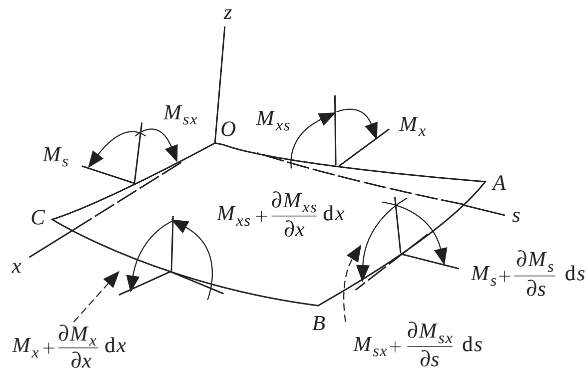


Figure 3. The moments acting on $OABC$. M_s, M_x, bending moments per unit length of axial section and a section perpendicular to the axis of a cylindrical shell, respectively; M_{xs}, M_{sx}, twisting moments per unit length of an axial section of a cylindrical shell.

Then the strain energy of $OABC$ due to the membrane stresses is the same as in plate theory and contains the three contributions:

$$\frac{1}{2}N_x\,\varepsilon_x\,dx\,ds=\frac{Eh}{2(1-\nu^2)}(\varepsilon_x+\nu\varepsilon_s)\epsilon_x\,dx\,ds,\quad \frac{1}{2}N_s\,\varepsilon_s\,dx\,ds=\frac{Eh}{2(1-\nu^2)}(\varepsilon_s+\nu\varepsilon_x)\epsilon_s\,dx\,ds,$$

$$\frac{1}{2}N_{xs}\,\gamma\,dx\,ds=\frac{Eh\gamma}{4(1+\nu)}\gamma\,dx\,ds, \tag{3}$$

while that due to the moments contains

$$-\tfrac{1}{2}M_x\,\kappa_x\,dx\,ds=-\tfrac{1}{2}[-D(\kappa_x+\nu\kappa_s)\kappa_x]\,dx\,ds,$$

$$-\tfrac{1}{2}M_s\,\kappa_s\,dx\,ds=-\tfrac{1}{2}[-D(\kappa_s+\nu\kappa_x)\kappa_s]\,dx\,ds,\qquad M_{xs}\,\chi_{xs}\,dx\,ds=D(1-\nu)\chi_{xs}^2\,dx\,ds. \tag{4}$$

The kinetic energy in $OABC$ is given by

$$\tfrac{1}{2}m(u_t^2+v_t^2+w_t^2)\,dx\,ds, \tag{5}$$

where m = mass density. However, Kraus [18] regards the contributions of kinetic energy by u and v to be insignificant, and therefore he neglects them in equation (5) and retains only

$$\tfrac{1}{2}mw_t^2\,dx\,ds \tag{6}$$

as the kinetic energy. Therefore the total energy of vibration at time t, from equations (3), (4) and (6), after substituting (1) and (2) therein, is

$$\begin{aligned}E(t)=\tfrac{1}{2}\iint\Bigg\{&m\left(\frac{\partial w}{\partial t}\right)^2+\frac{Eh}{1-\nu^2}\Bigg[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial s}+\frac{1}{R}w\right)^2\\&+2\nu\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial s}+\frac{1}{R}w\right)+\tfrac{1}{2}(1-\nu)\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial s}\right)^2\Bigg]\\&+D\Bigg[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial s^2}\right)^2+2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial s^2}+2(1-\nu)\left(\frac{\partial^2 w}{\partial x\,\partial s}\right)^2\Bigg]dx\,ds.\end{aligned} \tag{7}$$

Applying the principle of virtual work to equation (7), we obtain the following system of three PDEs:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\,\partial s} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial s^2} + \frac{\nu}{R}\frac{\partial w}{\partial x} = 0, \tag{8}$$

$$\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\,\partial s} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial s^2} + \frac{1}{R}\frac{\partial w}{\partial s} = 0, \tag{9}$$

$$\frac{h^2}{12}\Delta^2 w + \frac{1}{R}\left(\frac{1}{R}w + \frac{\partial v}{\partial s} + \nu\frac{\partial u}{\partial x}\right) = -m\frac{1-\nu^2}{Eh}\frac{\partial^2 w}{\partial t^2}. \tag{10}$$

Remark 2.1. Had we not omitted

$$\int \tfrac{1}{2} m(u_t^2 + v_t^2)\,\mathrm{d}x\,\mathrm{d}s$$

from equation (7), then equations (8) and (9) would contain inertial terms respectively as follows:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\,\partial s} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial s^2} + \frac{\nu}{R}\frac{\partial w}{\partial x} = m\frac{1-\nu^2}{Eh}\frac{\partial^2 u}{\partial t^2}, \tag{8$'$}$$

$$\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\,\partial s} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial s^2} + \frac{1}{R}\frac{\partial w}{\partial s} = m\frac{1-\nu^2}{Eh}\frac{\partial^2 v}{\partial t^2}. \tag{9$'$}$$

For the coupled system (8)′, (9)′ and (10), the kind of stabilization problem to be studied in section 4 is actually much easier than the one to be treated here. Therefore here we will treat only the model (8), (9) and (10).

Remark 2.2. The displacement w can be decoupled from u and v by Donnell's method [18], resulting in

$$\frac{h^2}{12}\Delta^4 w + \frac{1-\nu^2}{R^2}\frac{\partial^4 w}{\partial x^4} = -\frac{m(1-\nu^2)}{Eh}\frac{\partial^2}{\partial t^2}\Delta^2 w. \tag{11}$$

This says that the shell system (8–10) is essentially equivalent to a single scalar PDE of order 8.

3. DISSIPATIVE BOUNDARY CONDITIONS, FUNCTIONAL–ANALYTIC SETTING AND CONTRACTION SEMIGROUP OF EVOLUTION

First, we adjust the notation by writing x_1 and x_2 for x and s, respectively. Let $\Omega \in \mathbb{R}^2$ be the underlying co-ordinate domain for Donnell's shallow circular cylindrical shell; Ω is a bounded open domain with boundary $\partial\Omega$ which is C^2-smooth.

The strain energy derived in equations (3), (4) and (7) corresponds to a sesquilinear form

$$\mathrm{a}\left(\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix},\begin{bmatrix}u_2\\v_2\\w_2\end{bmatrix}\right)$$

$$\equiv K\int_\Omega \left\{u_{1x_1}\bar{u}_{2x_1} + \left\{v_{1x_2}+\frac{1}{R}w_1\right)\left(\bar{v}_{2x_2}+\frac{1}{R}\bar{w}_2\right)\right.$$

$$\left. + \nu\left[v_{1x_1}\left(\bar{v}_{2x_2}+\frac{1}{R}\bar{w}_2\right)+\left(v_{1x_2}+\frac{1}{R}w_1\right)\bar{u}_{2x_1}\right] + \frac{1-\nu}{2}(u_{1x_2}+v_{1x_1})(\bar{u}_{2x_2}+\bar{v}_{2x_1})\right\}\mathrm{d}x$$

$$+ D\int_D [(\Delta w_1)(\Delta\bar{w}_2) + (1-\nu)(2w_{1x_1x_2}\bar{w}_{2x_1x_2} - w_{1x_1x_1}\bar{w}_{2x_2x_2} - w_{1x_2x_2}\bar{w}_{2x_1x_1})]\,\mathrm{d}x, \tag{12}$$

where

$$K \equiv \frac{Eh}{1-\nu^2}, \qquad x = (x_1, x_2) \in \Omega, \qquad \mathrm{d}x = \mathrm{d}x_1\,\mathrm{d}x_2,$$

and subscripts involving x_1 and x_2 mean partial derivatives with respect to x_1 and x_2. Here the sesquilinear form is taken to be *complex* in preparation for the study in section 4.

The following is an easy generalization of the Rayleigh–Green formula for a plate.

Lemma 3.1. (integration by parts formula). Let (u_i, v_i, w_i), $i = 1, 2$, be sufficiently smooth on Ω. Then, for the sesquilinear form a defined in equation (12), we have

$$\mathrm{a}\left(\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix},\begin{bmatrix}u_2\\v_2\\w_2\end{bmatrix}\right) = \int_\Omega\left(\mathbb{A}\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}\right)\cdot\begin{bmatrix}\bar{u}_2\\\bar{v}_2\\\bar{w}_2\end{bmatrix}\mathrm{d}x$$

$$+\int_{\partial\Omega}\left\{\mathscr{B}_1(u_1, v_1, w_1)\cdot\bar{u}_2 + \mathscr{B}_2(u_1, v_1, w_1)\cdot\bar{v}_2\right.$$

$$\left. - (B_1 w_1)\bar{w}_2 + (B_2 w_1)\frac{\partial\bar{w}_2}{\partial n}\right]\mathrm{d}\sigma, \tag{13}$$

where $\mathbb{A}$, $\mathscr{B}_1$, $\mathscr{B}_2$, B_1 and B_2 are defined by

$$\mathbb{A}\begin{bmatrix}u\\v\\w\end{bmatrix} = -K\begin{bmatrix} u_{x_1x_1}+\frac{1+\nu}{2}v_{x_1x_2}+\frac{1-\nu}{2}u_{x_2x_2}+\frac{\nu}{R}w_{x_1}\\ \frac{1+\nu}{2}u_{x_1x_2}+\frac{1-\nu}{2}v_{x_1x_1}+v_{x_2x_2}+\frac{1}{R}w_{x_2}\\ -\left(\frac{h^2}{12}\Delta^2 w+\frac{1}{R}\left(\nu u_{x_1}+v_{x_2}+\frac{1}{R}w\right)\right)\end{bmatrix}, \qquad \text{on } \Omega, \tag{14}$$

$$\mathscr{B}_1(u, v, w) = K\left\{\left[u_{x_1} + v\left(v_{x_2} + \frac{1}{R}w\right)\right]n_1 + \frac{1-v}{2}(u_{x_2} + v_{x_1})n_2\right\}, \qquad \text{on } \partial\Omega,$$

$$\mathscr{B}_2(u, v, w) = K\left\{\left[vu_{x_1} + \left(v_{x_2} + \frac{1}{R}w\right)\right]n_2 + \frac{1-v}{2}(u_{x_2} + v_{x_1})n_1\right\}, \qquad \text{on } \partial\Omega,$$

$$B_1 w = D\left\{\frac{\partial}{\partial n}(\Delta w) - (1-v)\frac{\partial}{\partial\sigma}\left[n_1 n_2 (w_{x_1 x_1} - w_{x_2 x_2}) - (n_1^2 - n_2^2)w_{x_1 x_2}\right]\right\}, \qquad \text{on } \partial\Omega,$$

$$B_2 w = D\{v\Delta w + (1-v)[n_1^2 w_{x_1 x_1} + n_2^2 w_{x_2 x_2} + 2n_1 n_2 w_{x_1 x_2}]\}, \qquad \text{on } \partial\Omega,$$

with $n = (n_1, n_2)$, the unit outward normal, and $\partial/\partial\sigma = -n_2\,\partial/\partial x_1 + n_1\,\partial/\partial x_2$, the counterclockwise tangential derivative on $\partial\Omega$.

According to equation (7), the total energy of the system is given by

$$E = KE + PE_1 + PE_2,$$

where, apart from a factor of 1/2,

$$KE = m\int_\Omega |w_t(x, t)|^2\,\mathrm{d}x$$

$$PE_1 = PE_1(u, v, w; t)$$

$$= K\int_\Omega \left\{|u_{x_1}(x, t)|^2 + \left|v_{x_2}(x, t) + \frac{1}{R}w(x, t)\right|^2\right.$$

$$\left. + 2v\,\mathrm{Re}\,u_{x_1}(x, t)\left[\bar{v}_{x_2}(x, t) + \frac{1}{R}\bar{w}(x, t)\right] + \frac{1-v}{2}|u_{x_2}(x, t) + v_{x_1}(x, t)|^2\right\}\mathrm{d}x,$$

$$PE_2 = PE_2(w; t)$$

$$= D\int_\Omega \{|\Delta w(x, t)|^2 + 2(1-v)[|w_{x_1 x_2}(x, t)|^2 - \mathrm{Re}\,w_{x_1 x_1}(x, t)\bar{w}_{x_2 x_2}(x, t)]\}\,\mathrm{d}x.$$

Here, KE is the kinetic energy, PE_1, is the stretching strain energy and PE_2 is the bending strain energy.

In sections 3–4, we consider the case in which $\partial\Omega$ consists of two disconnected parts Γ_0 and Γ_1 :

$$\partial\Omega = \Gamma_0 \dot{\cup} \Gamma_1, \tag{15}$$

where each of Γ_0 and Γ_1 is non-empty and closed. (*A priori*, Ω cannot be a simply connected domain.) The case in which Γ_0 and Γ_1 have a non-empty intersection is treated in section 5, where we will require that Γ_0 and Γ_1 overlap only at their two end points which must also be corner points. On Γ_0, we assume that the shell is clamped:

$$\begin{bmatrix} u(x,t)=0, & v(x,t)=0, \\ w(x,t)=0, & \dfrac{\partial w(x,t)}{\partial n}=0, \\ \forall x\in\Gamma_0, & \forall t>0. \end{bmatrix} \tag{16}$$

The boundary conditions in equations (16) on Γ_0 are energy-conserving. We now derive linear dissipative boundary conditions on Γ_1 by differentiating the energy:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega m|w_t|^2\,\mathrm{d}x + \mathrm{a}\left(\begin{bmatrix}u\\v\\w\end{bmatrix},\begin{bmatrix}u\\v\\w\end{bmatrix}\right)\right\} \\
&= \mathrm{Re}\left\{2\int_\Omega m w_{tt}\,\bar{w}_t\,\mathrm{d}x + 2\mathrm{a}\left(\begin{bmatrix}u\\v\\w\end{bmatrix},\begin{bmatrix}u_t\\v_t\\w_t\end{bmatrix}\right)\right\} \\
&= \mathrm{Re}\left[2\int_\Omega m w_{tt}\,\bar{w}_t\,\mathrm{d}x + 2\mathrm{a}\int_\Omega\left(\mathbb{A}\begin{bmatrix}u\\v\\w\end{bmatrix}\right)\cdot\begin{bmatrix}\bar{u}_t\\\bar{v}_t\\\bar{w}_t\end{bmatrix}\mathrm{d}x, \qquad \text{(by Lemma 3.1)}\right. \\
&\left. \quad + 2\int_{\Gamma_1}\left\{\mathscr{B}_1(u,v,w)\cdot\bar{u}_t + \mathscr{B}_2(u,v,w)\cdot\bar{v}_t - (B_1 w)\bar{w}_t + (B_2 w)\frac{\partial\bar{w}_t}{\partial n}\right\}\mathrm{d}\sigma\right].\end{aligned} \tag{17}$$

On Ω, we know from section 2 that u, v and w satisfy the PDE system

$$\begin{bmatrix}0\\0\\m w_{tt}\end{bmatrix} + \mathbb{A}\begin{bmatrix}u\\v\\w\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}. \tag{18}$$

Therefore, the two integrals on Ω in equation (17) add to zero. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E = 2\int_{\Gamma_1}\left\{\mathscr{B}_1(u,v,w)\cdot\bar{u}_t + \mathscr{B}_2(u,v,w)\bar{v}_t - (B_1 w)\bar{w}_t + (B_2 w)\frac{\partial\bar{w}_t}{\partial n}\right\}\mathrm{d}\sigma. \tag{19}$$

On Γ_1, if

$$\left[\begin{array}{l}\mathscr{B}_1(u,v,w)=0,\\ \mathscr{B}_2(u,v,w)=0,\\ \begin{bmatrix}B_1 w\\ -B_2 w\end{bmatrix} = F\begin{bmatrix}w_t\\ \dfrac{\partial w_t}{\partial n}\end{bmatrix}, \qquad \forall t>0, \quad \text{on } \Gamma_1,\end{array}\right. \tag{20}$$

where the feedback gain matrix F is a symmetric positive semidefinite matrix, with sufficiently smooth (say, in $C^2(\Gamma_1)$) entries of the form

$$F = \begin{bmatrix} d_1(\cdot) & \beta(\cdot) \\ \beta(\cdot) & d_2(\cdot) \end{bmatrix}, \qquad d_1(x) \geqslant 0,\, d_2(x) \geqslant 0,\, \beta(x) \in \mathbb{R},$$

$$d_1(x) d_2(x) - \beta^2(x) \geqslant 0, \quad x \in \Gamma_1, \tag{21}$$

then equations (19) and (20) give

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -2 \int_{\Gamma_1} \left[w_t, \frac{\partial w_t}{\partial n} \right] F \begin{bmatrix} \bar{w}_t \\ \dfrac{\partial \bar{w}_t}{\partial n} \end{bmatrix} \mathrm{d}\sigma \leqslant 0.$$

Therefore the boundary conditions in equation (20) on Γ_1 cause energy dissipation.

There are many types of boundary conditions other than equation (20) that may also cause energy dissipation on Γ_1. However, so far equation (20) is the only major type for which we are able to establish the uniform exponential decay result in this paper.

Let $H^s(\Omega)$ $(= W^{s,2}(\Omega))$ be the usual Sobolev space of order $s \geqslant 0$ on Ω and, for a positive integer k, let

$$H^k_{\Gamma_0}(\Omega) = \left\{ f \in H^k(\Omega) \,|\, f = \frac{\partial f}{\partial n} = \cdots = \frac{\partial^{k-1} f}{\partial n^{k-1}} = 0 \text{ on } \Gamma_0 \right\}.$$

We will often write $H^k_{\Gamma_0}$ instead of $H^k_{\Gamma_0}(\Omega)$ for brevity.

Let us decompose the bilinear form a in equation (12) into the following associated bilinear and linear forms. First, we define reduced sesquilinear forms

$$\mathrm{a}_I\left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right) = K \int_\Omega [u_{1x_1} \bar{u}_{2x_1} + v_{1x_2} \bar{v}_{2x_2} + \nu (u_{1x_1} \bar{u}_{2x_2} + v_{1x_2} \bar{u}_{2x_1})$$

$$+ \frac{1-\nu}{2} (u_{1x_2} + v_{1x_1})(\bar{u}_{2x_2} + \bar{v}_{2x_1})] \,\mathrm{d}x, \quad (u_i, v_i) \in H^1_{\Gamma_0} \times H^1_{\Gamma_0}, \quad i = 1, 2, \tag{22}$$

$$\mathrm{a}_{II}(w_1, w_2) = D \int_\Omega \bigg[(\varDelta w_1)(\varDelta \bar{w}_2) + (1-\nu)(2 w_{1x_1x_2} \bar{w}_{2x_1x_2} - w_{1x_1x_1} \bar{w}_{2x_2x_2}$$

$$- w_{1x_2x_2} \bar{w}_{2x_1x_1}) + \frac{1}{R^2} w_1 \bar{w}_2 \bigg] \mathrm{d}x, \qquad w_1, w_2 \in H^2_{\Gamma_0}, \quad i = 1, 2. \tag{23}$$

Next, for given $w \in H^2_{\Gamma_0}$, define the (conjugate) linear form

$$\theta^w_I\left(\begin{bmatrix} u \\ v \end{bmatrix} \right) = K \int_\Omega \frac{1}{R} w (\nu \bar{u}_{x_1} + \bar{v}_{x_2}) \,\mathrm{d}x, \qquad (u, v) \in H^1_{\Gamma_0} \times H^1_{\Gamma_0}, \tag{24}$$

and, for given $(u, v) \in H^1_{\Gamma_0} \times H^1_{\Gamma_0}$, define the (conjugate) linear form

$$\theta^{(u,v)}_{II}(w) = K \int_\Omega \frac{1}{R} (\nu u_{x_1} + v_{x_2}) \bar{w} \,\mathrm{d}x, \qquad w \in H^2_{\Gamma_0}. \tag{25}$$

Then it is easy to check that

$$\mathrm{a}\left(\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix}\right) = \mathrm{a}_I\left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right) + \mathrm{a}_{II}(w_1, w_2)$$

$$+ \theta_I^{w_1}\left(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right) + \theta_{II}^{(u_1,v_1)}(w_2). \tag{26}$$

Lemma 3.2. Let θ_I and θ_{II} be defined as in equations (24) and (15). Then:

(i) θ_I is linear with respect to w and conjugate linear with respect to (u, v); i.e.,

$$\theta_I^{cw_1 + w_2}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = c\theta_I^{w_1}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) + \theta_I^{w_2}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right),$$

$$\theta_I^{w}\left(c\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right) = \bar{c}\theta_I^{w}\left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\right) + \theta_I^{w}\left(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right),$$

for all $c \in \mathbb{C}$; w, w_1, $w_2 \in H^2_{\Gamma_0}$; and (u, v), (u_1, v_1), $(u_2, v_2) \in H^1_{\Gamma_0} \times H^1_{\Gamma_0}$. For any given $w \in H^2_{\Gamma_0}$, $\theta_I^w(\cdot)$ is a continuous (conjugate) linear functional on $H^1_{\Gamma_0} \times H^1_{\Gamma_0}$.

(*ii*) θ_{II} is linear with respect to (u, v) and conjugate linear with respect to w; i.e.,

$$\theta_{II}^{c(u_1,v_1) + (u_2,v_2)}(w) = c\theta_{II}^{(u_1,v_1)}(w) + \theta_{II}^{(u_2,v_2)}(w), \tag{27}$$

$$\theta_{II}^{(u,v)}(cw_1 + w_2) = \bar{c}\theta_{II}^{(u,v)}(w_1) + \theta_{II}^{(u,v)}(w_2), \tag{28}$$

for all $c \in \mathbb{C}$; (u, v), (u_1, v_1), $(u_2, v_2) \in H^1_{\Gamma_0} \times H^1_{\Gamma_0}$; and w, w_1, $w_2 \in H^2_{\Gamma_0}$. For any given $(u, v) \in H^1_{\Gamma_0} \times H^1_{\Gamma_0}$, $\theta_{II}^{(u,v)}(\cdot)$ is a continuous conjugate linear functional on $H^2_{\Gamma_0}$.

Proof. The proof is obvious

Lemma 3.3. Define a mapping

$$\mathscr{L}: H^2_{\Gamma_0} \to H^1_{\Gamma_0} \times H^1_{\Gamma_0}$$

by

$$\mathscr{L}(\tilde{w}) = (\tilde{u}, \tilde{v}), \qquad \tilde{w} \in H^2_{\Gamma_0}, \qquad (\tilde{u}, \tilde{v}) \in H^1_{\Gamma_0} \times H^1_{\Gamma_0},$$

where $(\tilde{u}, \tilde{v})$ is the unique solution to the variational problem

$$\mathrm{a}_I\left(\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}\right) = -\theta_I^{\tilde{w}}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right), \qquad \forall (u, v) \in H^1_{\Gamma_0} \times H^1_{\Gamma_0}. \tag{29}$$

Then $\mathscr{L}$ is a continuous linear transformation from $H^2_{\Gamma_0}$ into $H^1_{\Gamma_0} \times H^1_{\Gamma_0}$.

Proof. The bilinear form a_I is coercive on $H^1_{\Gamma_0} \times H^1_{\Gamma_0}$ because of (a version of) Korn's Lemma. Therefore the variational problem (29) has a unique solution $(\tilde{u}, \tilde{v}) \in H^1_{\Gamma_0} \times H^1_{\Gamma_0}$. The rest of the proof follows from the Riesz Representation Theorem and Lemma 3.2.

The solution $(\tilde{u}, \tilde{v}) = \mathscr{L}\tilde{w}$ actually has higher regularity than $H^1_{\Gamma_0} \times H^1_{\Gamma_0}$ as promised by Lemma 3.3. This is given in the following

Corollary 3.4. For given $\tilde{w} \in H^2_{\Gamma_0}$, let $(\tilde{u}, \tilde{v}) = \mathscr{L}\tilde{w}$. Then

$$(\tilde{u}, \tilde{v}) \in (H^1_{\Gamma_0} \times H^1_{\Gamma_0}) \cap (H^3(\Omega) \times H^3(\Omega)),$$

and $(\tilde{u}, \tilde{v})$ satisfies

$$\mathrm{a}_I\left(\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}\right) + \theta_I^{\tilde{w}}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) - \int_{\Gamma_0} \{\mathscr{B}_1(u, v, w) \cdot \mathscr{B}_2(u, v, w) \cdot \bar{v}\}\, \mathrm{d}\sigma = 0,$$

$$\forall (u, v) \in H^1(\Omega) \times H^1(\Omega).$$

Proof. Since $(\tilde{u}, \tilde{v}, \tilde{w})$ satisfies

$$\mathrm{a}_I\left(\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}\right) = -\theta_I^{\tilde{w}}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right), \qquad \forall (u, v) \in H^1_{\Gamma_0} \times H^1_{\Gamma_0}.$$

by calculus of variations we see that $(\tilde{u}, \tilde{v})$ forms a weak solution of the following elliptic system:

$$K\left(\tilde{u}_{x_1 x_1} + \frac{1+\nu}{2}\tilde{v}_{x_1 x_2} + \frac{1-\nu}{2}\tilde{u}_{x_2 x_2}\right) = -K \cdot \frac{\nu}{R}\tilde{w}_{x_1}, \qquad \text{on } \Omega, \tag{30}$$

$$K\left(\frac{1+\nu}{2}\tilde{u}_{x_1 x_2} + \frac{1-\nu}{2}\tilde{v}_{x_1 x_1} + \tilde{v}_{x_2 x_2}\right) = -K \cdot \frac{1}{R}\tilde{w}_{x_2}, \qquad \text{on } \Omega, \tag{31}$$

$$K\left[(\tilde{u}_{x_1} + \nu\tilde{v}_{x_2})n_2 + \frac{1-\nu}{2}(\tilde{u}_{x_2} + \tilde{v}_{x_1})n_1\right] = -K \cdot \frac{\nu}{R} n_1 \tilde{w}, \qquad \text{on } \Gamma_1, \tag{32}$$

$$K\left[(\nu\tilde{u}_{x_1} + \tilde{v}_{x_2})n_2 + \frac{1-\nu}{2}(\tilde{u}_{x_2} + \tilde{v}_{x_1})n_1\right] = -K \cdot \frac{1}{R} n_2 \tilde{w}, \qquad \text{on } \Gamma_1, \tag{33}$$

$$\tilde{u} = 0, \quad \tilde{v} = 0, \qquad \text{on } \Gamma_0.$$

The above system satisfies the conditions in Douglas and Nirenberg [22], and the classical elliptic regularity results of solutions hold. Since the RHS of equations (30) and (31) belong to $H^1(\Omega)$, and the RHS of equations (32) and (33) belong to $H^{3/2}(\Gamma_1)$, we obtain $(u, v) \in H^3(\Omega) \times H^3(\Omega)$.

The rest follows from integration by parts.

We are now in a position to determine the infinitesimal generator corresponding to the evolution equation (18) subject to boundary conditions (16) and (20), and with certain initial condition (u_0, v_0, w_0). We will take advantage of a natural tri-space setting $V \hookrightarrow H \hookrightarrow V^*$. Let

$$V \equiv H^2_{\Gamma_0}, \qquad H = L^2(\Omega), \qquad V^* = \text{the dual of } V \text{ pivotal to } H. \tag{34}$$

For any $w_1, w_2 \in V$, define

$$\langle\!\langle w_1, w_2 \rangle\!\rangle = a_{II}(w_1, w_2) + \theta_{II}^{\mathscr{L}w_1}(w_2), \qquad w_1, w_2 \in V. \tag{35}$$

Theorem 3.5. Expression (35) defines a symmetric positive-definite continuous sesquilinear form on V. Consequently, $\langle\!\langle\ ,\ \rangle\!\rangle$ constitutes an inner product for the Hilbert space V.

Proof. It is routine to check that for any $w \in H^2_{\Gamma_0}(\Omega)$, the functionals

$$\langle\!\langle \cdot, w \rangle\!\rangle = \mathrm{a}_{II}(\cdot, w) + \theta_{II}^{\mathscr{L}\cdot}(w) \colon H^2_{\Gamma_0}(\Omega) \to \mathbb{R},$$

and

$$\langle\!\langle w, \cdot \rangle\!\rangle = \mathrm{a}_{II}(w, \cdot) + \theta_{II}^{\mathscr{L}w}(\cdot) \colon H^2_{\Gamma_0}(\Omega) \to \mathbb{R},$$

are continuous. We now verify that, for any $w, w_1, w_2 \in H^2_{\Gamma_0}(\Omega)$ and any $c \in \mathbb{C}$,

$$\langle\!\langle w_1 + cw_2, w \rangle\!\rangle = \langle\!\langle w_1, w \rangle\!\rangle + c\langle\!\langle w_2, w \rangle\!\rangle. \tag{36}$$

By definition from equation (35), and by Lemma 3.2,

$$\begin{aligned}\langle\!\langle w_1 + cw_2, w \rangle\!\rangle &= \mathrm{a}_{II}(w_1 + cw_2, w) + \theta_{II}^{\mathscr{L}(w_1 + cw_2)}(w) \\ &= [\mathrm{a}_{II}(w_1, w) + \theta_{II}^{\mathscr{L}w_1}(w)] + [c\mathrm{a}_{II}(w_2, w) + c\theta_{II}^{\mathscr{L}w_2}(w)] \\ &= \langle\!\langle w_1, w \rangle\!\rangle + c\langle\!\langle w_2, w \rangle\!\rangle.\end{aligned}$$

Therefore, equation (36) has been verified. Similarly, we can also show that

$$\langle\!\langle w, w_1 + cw_2 \rangle\!\rangle = \langle\!\langle w, w_1 \rangle\!\rangle + \bar{c}\langle\!\langle w, w_2 \rangle\!\rangle.$$

Hence, $\langle\!\langle\ ,\ \rangle\!\rangle$ is a continuous sesquilinear form on V.

To show that $\langle\!\langle\ ,\ \rangle\!\rangle$ is symmetric, let $w_1, w_2 \in H^2_{\Gamma_0}$ and let $(u_i, v_i) = \mathscr{L}w_i$, $i = 1, 2$. Then

$$\begin{aligned}\langle\!\langle w_1, w_2 \rangle\!\rangle &= \mathrm{a}_{II}(w_1, w_2) + \theta_{II}^{\mathscr{L}w_1}(w_2) \qquad (37)\\ &= \mathrm{a}_{II}(w_1, w_2) \\ &\quad + \theta_{II}^{\mathscr{L}w_1}(w_2) + \left[\mathrm{a}_I\left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right) + \theta_I^{w_1}\left(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right)\right] \qquad \text{(by Lemma 3.3)} \\ &= \mathrm{a}\left(\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix}\right) \qquad \text{(by equation (26))} \qquad (38) \\ &= \overline{\mathrm{a}_{II}(w_2, w_1)} + \overline{\theta_{II}^{\mathscr{L}w_2}(w_1)} \\ &\quad + \left[\overline{\mathrm{a}_I\left(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\right)} + \overline{\theta_I^{w_2}\left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\right)}\right] \qquad \text{(by equation (29))} \\ &= \overline{\mathrm{a}_{II}(w_2, w_1)} + \overline{\theta_{II}^{\mathscr{L}w_2}(w_1)} \qquad \text{(by Lemma 3.3)} \\ &= \overline{\langle\!\langle w_2, w_1 \rangle\!\rangle}. \qquad (39)\end{aligned}$$

Finally, to show that $\langle\!\langle\ ,\ \rangle\!\rangle$ is positive definite, we use equations (37) and (38) by letting $w_1 = w_2 = w$ and $(u, v) = \mathscr{L}w$ therein, yielding

$$\langle\!\langle w, w \rangle\!\rangle = \mathrm{a}\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}, \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) > 0, \qquad \text{if } w \neq 0.$$

From Poincaré's inequality, we also have

$$\langle\!\langle w, w \rangle\!\rangle \geqslant \delta \|w\|_V^2, \qquad \text{for some } \delta > 0.$$

The proof is complete.

The sesquilinear form $\langle\!\langle \cdot , \cdot \rangle\!\rangle$ induces an operator A, the canonical isomorphism, on V:

$$\begin{cases} A \colon V \to V^*, \\ \langle\!\langle w_1, w_2 \rangle\!\rangle = \langle A w_1, w_2 \rangle_{V^* \times V}. \end{cases} \tag{40}$$

Lemma 3.6. (Integration by parts formula for the sesquilinear form $\langle\!\langle \cdot , \cdot \rangle\!\rangle$). Let $w_1, w_2 \in V$, and $(u_i, v_i) = \mathscr{L} w_i$ for $i = 1, 2$. Then

$$\langle\!\langle w_1, w_2 \rangle\!\rangle = \mathrm{a}\left(\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} \right), \tag{41}$$

and if w_1 and w_2 are sufficiently smooth, we have

$$\langle\!\langle w_1, w_2 \rangle\!\rangle = \int_\Omega \left\{ D\Delta^2 w_1 + K\left[\frac{1}{R}(\nu u_{1,x_1} + v_{1,x_2}) + \frac{1}{R^2} w_1 \right] \right\} \bar{w}_2 \, \mathrm{d}x$$

$$- \int_{\Gamma_1} \left[(B_1 w_1) \bar{w}_2 - (B_2 w_1) \frac{\partial \bar{w}_2}{\partial n} \right] \mathrm{d}\sigma. \tag{42}$$

Furthermore, if $w_1 \in V$, $(u_1, v_1) = \mathscr{L} w_1$, and $w_2 \in H^2(\Omega)$, then

$$\mathrm{a}_{II}(w_1, w_2) + \theta_{II}^{\mathscr{L} w_1}(w_2) = \int_\Omega \left\{ D\Delta^2 w + K\left[\frac{1}{R}(\nu u_{1,x_1} + v_{1,x_2}) + \frac{1}{R^2} w_1 \right] \right\} \bar{w}_2 \, \mathrm{d}x$$

$$- \int_{\partial\Omega} \left[(B_1 w_1) \bar{w}_2 - (B_2 w_1) \frac{\partial \bar{w}_2}{\partial n} \right] \mathrm{d}\sigma.$$

Proof. Equality (41) follows from equations (37) and (38) in the proof of Theorem 3.5.

From equation (41), we can apply Lemma 3.1 provided that (u_i, v_i, w_i), $i = 1, 2$, are sufficiently smooth. But (u_i, v_i), $i = 1, 2$, will always be smooth enough, by Corollary 3.4. Therefore we only need w_1 and w_2 to be sufficiently smooth, and we obtain

$$\langle\!\langle w_1, w_2 \rangle\!\rangle = \mathrm{a}\left(\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} \right)$$

$$= \int_\Omega \left(\mathbb{A} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} \right) \cdot \begin{bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{w}_2 \end{bmatrix} \mathrm{d}x + \int_{\Gamma_1} [\mathscr{B}_1(u_1, v_1, w_1) \cdot \bar{u}_2 + \mathscr{B}_2(u_1, v_1, w_1) \cdot \bar{v}_2] \, \mathrm{d}\sigma$$

$$- \int_{\Gamma_1} \left[(B_1 w_1) \cdot \bar{w}_2 - (B_2 w_1) \frac{\partial \bar{w}_2}{\partial n} \right] \mathrm{d}\sigma.$$

But, by the proof of Corollary 3.4, equations (30)–(33) hold, and so the first two components of $\mathbb{A}(u_1, v_1, w_1)^{\mathrm{T}}$ are zero, and $\mathscr{B}_1(u_1, v_1, w_1) = 0$, $\mathscr{B}_2(u_2, v_2, w_2) = 0$ on Γ_1. Hence equation (42) follows.

For $v_1, v_2 \in V$, we also define another sesquilinear form, cf. (21)

$$\mathbb{b}(v_1, v_2) = \int_{\Gamma_1} \left[v_1, \frac{\partial v_1}{\partial n} \right] F \begin{bmatrix} \bar{v}_2 \\ \dfrac{\partial \bar{v}_2}{\partial n} \end{bmatrix} \mathrm{d}\sigma. \tag{43}$$

Then this sesquilinear form $\mathbb{b}$ induces an operator B:

$$\begin{cases} B\colon V \to V^*, \\ \mathbb{b}(v_1, v_2) = \langle Bv_1, v_2 \rangle_{V^* \times V}. \end{cases} \tag{44}$$

Obviously, B is a symmetric positive semidefinite operator:

$$\langle Bv, v \rangle \geqslant 0, \qquad \forall v \in V.$$

Let $\mathscr{H} \equiv V \times H$ be the Hilbert space equipped with the inner product

$$\left\langle \begin{bmatrix} w_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ z_2 \end{bmatrix} \right\rangle_{\mathscr{H}} \equiv \langle\!\langle w_1, w_2 \rangle\!\rangle + \langle z_1, z_2 \rangle_H, \qquad (w_i, z_i) \in \mathscr{H}, \quad i = 1, 2. \tag{45}$$

The associated norm will be denoted by $\| \ \|_{\mathscr{H}}$, or briefly $\| \ \|$, in case no ambiguity should occur.

We now define an operator $\mathscr{A}$ on $\mathscr{H}$ by

$$\mathscr{A} = \begin{bmatrix} 0 & I \\ -A & -B \end{bmatrix} \colon D(\mathscr{A}) \to \mathscr{H}, \tag{46}$$

$$D(\mathscr{A}) = \left\{ \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \middle| \, w_0, w_1 \in V, \quad -(Aw_0 + Bw_1) \in H \right\}. \tag{47}$$

Lemma 3.7. The operator $\mathscr{A}$ is dissipative on $\mathscr{H}$.

Proof. Let $(w, z) \in D(\mathscr{A})$. Then

$$\begin{aligned} \operatorname{Re} \left\langle \mathscr{A} \begin{bmatrix} w \\ z \end{bmatrix}, \begin{bmatrix} w \\ z \end{bmatrix} \right\rangle_{\mathscr{H}} &= \operatorname{Re} \left\langle \begin{bmatrix} z \\ -(Aw + Bz) \end{bmatrix}, \begin{bmatrix} w \\ z \end{bmatrix} \right\rangle_{\mathscr{H}} \\ &= \operatorname{Re} [\langle\!\langle z, w \rangle\!\rangle + \langle Aw - Bz, z \rangle_{L^2(\Omega)}] \\ &= \operatorname{Re} [\langle\!\langle z, w \rangle\!\rangle - \langle\!\langle w, z \rangle\!\rangle - \langle Bz, z \rangle_{V^* \times V}] \\ &= -\langle Bz, z \rangle_{V^* \times V} \\ &\leqslant 0. \end{aligned}$$

Theorem 3.8. $\mathscr{A}$ is the infinitesimal generator of a C_0-semigroup of contractions in $\mathscr{H}$.

Proof. We know that $D(\mathscr{A})$ is dense in $\mathscr{H}$ because $\mathscr{A}^{-1}$ is a bounded operator on $\mathscr{H}$ and $\mathscr{A}$ is dissipative.

We may now apply the Lumer–Phillips Theorem [23]. We show that, for some $\lambda_0 > 0$,

$$\operatorname{Range} (\lambda_0 I - \mathscr{A}) = \mathscr{H}. \tag{48}$$

It is easy to check that equation (48) is equivalent to

$$\text{Range}\,(A + \lambda_0^2 I + \lambda_0 B) = V^*. \tag{49}$$

Choose any $\lambda_0 > 0$. Then

$$\langle (A + \lambda_0^2 I + \lambda_0 B) w_1, w_2 \rangle_{V^* \times V} = \langle\!\langle w_1, w_2 \rangle\!\rangle + \lambda_0^2 \int_\Omega w_1 \bar{w}_2 \, \mathrm{d}x$$

$$+ \lambda_0 \int_{\Gamma_1} \left[w_1, \frac{\partial w_1}{\partial n} \right] F \begin{bmatrix} \bar{w}_2 \\ \dfrac{\partial \bar{w}_2}{\partial n} \end{bmatrix} \mathrm{d}\sigma,$$

and the RHS above becomes a bounded coercive sesquilinear form on $H^2_{\Gamma_0}(\Omega)$. We therefore have equation (49) by the Lax–Milgram Theorem.

Incidentally, we note from the above that $\lambda_0 = 0$ belongs to $\rho(\mathscr{A})$, the resolvent set of $\mathscr{A}$; i.e., $\mathscr{A}^{-1}$ is a bounded linear operator on $\mathscr{H}$.

The following regularity result is familiar (Lagnese [1], Pazy [23]).

Corollary 3.9. Let $\mathscr{A}$ be the infinitesimal generator of C_0-semigroup $S(t)$, $t > 0$, on $\mathscr{H}$ as in Theorem 3.8. Then, for the Cauchy problem,

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} w(\cdot\,, t) \\ \dot{w}(\cdot\,, t) \end{bmatrix} = \mathscr{A} \begin{bmatrix} w(\cdot\,, t) \\ \dot{w}(\cdot\,, t) \end{bmatrix}, \qquad t > 0,$$

$$\begin{bmatrix} w(\cdot\,, 0) \\ \dot{w}(\cdot\,, 0) \end{bmatrix} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in \mathscr{H}, \tag{50}$$

we have $(w, \dot{w}) = S(\cdot)\,(w_0, w_1) \in C^0([0, \infty); \mathscr{H})$. Furthermore, if $(w_0, w_1) \in D(\mathscr{A})$, then w satisfies

$$\begin{cases} w \in C^1([0, \infty); V) \cap C^2([0, \infty); H), \\ Aw + B\dot{w} \in C^0([0, \infty); H), \\ \ddot{w} + Aw + B\dot{w} = 0, \qquad t > 0. \end{cases} \tag{51}$$

Note that we have tacitly set $m = 1$ (through normalization) in equation (45) so that the functional differential equation (51) is just the PDE

$$w_{tt} + D\Delta^2 w + K\left[\frac{1}{R} (v u_{x_1} + v_{x_2}) + \frac{1}{R^2} w \right] = 0 \tag{52}$$

as obtained from the third component of the system (18) by setting $m = 1$ therein, subject to the boundary conditions (16) and (20). For the rest of the paper, we will continue to use $m = 1$ for convenience.

4. A FREQUENCY DOMAIN METHOD WITH ENERGY MULTIPLIERS FOR PROVING UNIFORM EXPONENTIAL DECAY OF ENERGY

We first recall the following "Frequency Domain Theorem" (Huang [24], Prüss [25]) for proving exponential decay.

Theorem 4.1. Let $e^{\mathscr{A}t}$, $t > 0$, be a C_0-semigroup in a Hilbert space satisfying $\|e^{\mathscr{A}t}\| < M$, $\forall t \geqslant 0$. Then $e^{\mathscr{A}t}$ decays exponentially if and only if:

$$\text{(i)} \quad \{i\omega \mid \omega \in \mathbb{R}\} \subset \rho(\mathscr{A}), \quad (\rho(\mathscr{A}) \text{ is the resolvent set of } \mathscr{A}), \tag{53}$$

$$\text{(ii)} \quad \sup\{\|(i\omega I - \mathscr{A})^{-1}\| \mid \omega \in \mathbb{R}|\} < \infty. \tag{54}$$

Lemma 4.2. (energy identity). Let $w \in H^{7/2+\varepsilon}(\Omega)$, and $u, v \in H^{3/2+\varepsilon}(\Omega)$ for some small $\varepsilon > 0$. Assume that $w = \partial w/\partial n = 0$ and $u = v = 0$ on Γ_0. Then

$$\operatorname{Re} \int_\Omega \mathbb{A} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \cdot \begin{bmatrix} x \cdot \nabla \bar{u} \\ x \cdot \nabla \bar{v} \\ x \cdot \nabla \bar{w} \end{bmatrix} \mathrm{d}x = \mathscr{T}_0 + \sum_{j=1}^{9} \mathscr{T}_j, \tag{55}$$

where

$$\mathscr{T}_0 \equiv a_{II}(w, w) + \operatorname{Re} \theta_{II}^{(u,v)}(w),$$

$$\mathscr{T}_1 \equiv -D \int_{\Gamma_0} (x \cdot n) \left|\frac{\partial^2 w}{\partial n^2}\right|^2 \mathrm{d}\sigma, \qquad \mathscr{T}_2 \equiv \operatorname{Re} \int_{\Gamma_1} (B_1 w)(x \cdot \nabla \bar{w}) \,\mathrm{d}\sigma,$$

$$\mathscr{T}_3 \equiv -\operatorname{Re} \int_{\Gamma_1} (B_2 w) \frac{\partial}{\partial n} (x \cdot \nabla \bar{w}) \,\mathrm{d}\sigma,$$

$$\mathscr{T}_4 \equiv -K \int_{\Gamma_0} (x \cdot n) \left[|u_{x_1}|^2 + |v_{x_2}|^2 + 2\nu \operatorname{Re} u_{x_1} \bar{v}_{x_2} + \frac{1-\nu}{2} |u_{x_2} + v_{x_1}|^2 \right] \mathrm{d}\sigma,$$

$$\mathscr{T}_5 \equiv -\operatorname{Re} \int_{\Gamma_1} [\mathscr{B}_1(u, v, w)(x \cdot \nabla \bar{u}) + \mathscr{B}_2(u, v, w)(x \cdot \nabla \bar{v})] \,\mathrm{d}\sigma,$$

$$\mathscr{T}_6 \equiv \frac{D}{2} \int_{\Gamma_0} (x \cdot n) \left|\frac{\partial^2 w}{\partial n^2}\right|^2 \mathrm{d}\sigma,$$

$$\mathscr{T}_7 \equiv \frac{D}{2} \int_{\Gamma_1} (x \cdot n) [|\Delta w|^2 + 2(1-\nu)(|w_{x_1 x_2}|^2 - \operatorname{Re} w_{x_1 x_1} \bar{w}_{x_2 x_2})] \,\mathrm{d}\sigma,$$

$$\mathscr{T}_8 \equiv \frac{K}{2} \int_{\Gamma_0} (x \cdot n) \left[|u_{x_1}|^2 + |v_{x_2}|^2 + 2\nu \operatorname{Re} u_{x_1} \bar{v}_{x_2} + \frac{1-\nu}{2} |u_{x_2} + v_{x_1}|^2 \right] \mathrm{d}\sigma,$$

$$\mathscr{T}_9 \equiv \frac{K}{2}\int_{\Gamma_1}(x\cdot n)\left[|u_{x_1}|^2+|v_{x_2}|^2+2\nu\,\mathrm{Re}\,u_{x_1}\bar{v}_{x_2}+\frac{1-\nu}{2}|u_{x_2}+v_{x_1}|^2\right]\mathrm{d}\sigma,$$

In particular, if $(u, v) = \mathscr{L}w$, then

$$\mathscr{T}_0 = \langle\langle w, w\rangle\rangle \qquad \text{and} \qquad \mathscr{T}_5 = 0.$$

Proof. From Lemma 3.1, we have

$$\int_\Omega \mathbb{A}\begin{bmatrix}u\\v\\w\end{bmatrix}\cdot\begin{bmatrix}x\cdot\nabla\bar{u}\\x\cdot\nabla\bar{v}\\x\cdot\nabla\bar{w}\end{bmatrix}\mathrm{d}x$$

$$=\mathrm{a}\left(\begin{bmatrix}u\\v\\w\end{bmatrix},\begin{bmatrix}x\cdot\nabla u\\x\cdot\nabla v\\x\cdot\nabla w\end{bmatrix}\right)$$

$$+\int_{\partial\Omega}\left[(B_1 w)(x\cdot\nabla\bar{w})-(B_2 w)\frac{\partial}{\partial n}(x\cdot\nabla\bar{w})\right]\mathrm{d}\sigma$$

$$+\int_{\partial\Omega}[\mathscr{B}_1(u,v,w)\cdot(x\cdot\nabla\bar{u})+\mathscr{B}_2(u,v,w)\cdot(x\cdot\nabla\bar{v})]\,\mathrm{d}\sigma. \tag{56}$$

Since $w = \partial w/\partial n = 0$ on Γ_0, we have $\nabla w = 0$ on Γ_0, and thus

$$B_2 w = D\Delta w = D\frac{\partial^2 w}{\partial n^2}, \quad \frac{\partial}{\partial n}(x\cdot\nabla w) = (x\cdot n)\Delta w = (x\cdot n)\frac{\partial^2 w}{\partial n^2}, \qquad \text{cf. (reference [26]).}$$

Therefore

$$\int_{\partial\Omega}\left[(B_1 w)(x\cdot\nabla\bar{w})-(B_2 w)\frac{\partial}{\partial n}(x\cdot\nabla\bar{w})\right]\mathrm{d}\sigma = \mathscr{T}_1+\mathscr{T}_2+T_3.$$

Also, $u = v = 0$ on Γ_0 gives $\nabla u = (\partial u/\partial n)n$, $\nabla v = (\partial v/\partial n)n$ on Γ_0, implying that

$$n_2 u_{x_1} = n_1 u_{x_2}, \quad n_2 v_{x_1} = n_1 v_{x_2}, \qquad \text{on } \Gamma_0. \tag{57}$$

These, along with $w = 0$ on Γ_0, give

$$\frac{1}{K}(\mathscr{B}_1(u,v,w))(x\cdot\nabla\bar{u}) = \left\{(u_{x_1}+\nu v_{x_2})n_1+\frac{1-\nu}{2}(u_{x_2}+v_{x_1})n_2\right\}(x_1\bar{u}_{x_1}+x_2\bar{u}_{x_2})$$

$$=(x\cdot n)\left[|u_{x_1}|^2+\nu\bar{u}_{x_1}v_{x_2}+\frac{1-\nu}{2}\bar{u}_{x_2}v_{x_1}+\frac{1-\nu}{2}|u_{x_2}|^2\right],$$

and

$$\frac{1}{K}(\mathscr{B}_2(u,v,w))(x\cdot\nabla\bar{v}) = \left\{(\nu u_{x_1}+v_{x_2})n_2 + \frac{1-\nu}{2}(u_{x_2}+v_{x_1})n_1\right\}(x_1\bar{v}_{x_1}+x_2\bar{v}_{x_2})$$

$$= (x\cdot n)\left[\nu u_{x_1}\bar{v}_{x_2} + |v_{x_2}|^2 + \frac{1-\nu}{2}\bar{u}_{x_2}\bar{v}_{x_1} + \frac{1-\nu}{2}|v_{x_1}|^2\right],$$

which correspond to $\mathscr{T}_4$; $\mathscr{T}_5$ is just the remainder part of that integral on Γ_1.

Now let us treat the first term on the RHS of equation (56). By equation (26), we have

$$\mathrm{a}\left(\begin{bmatrix}u\\v\\w\end{bmatrix},\begin{bmatrix}x\cdot\nabla u\\x\cdot\nabla v\\x\cdot\nabla w\end{bmatrix}\right) = \mathrm{a}_{II}(w, x\cdot\nabla w) + \mathrm{a}_I\left(\begin{bmatrix}u\\v\end{bmatrix},\begin{bmatrix}x\cdot\nabla u\\x\cdot\nabla v\end{bmatrix}\right) + \theta_I^w\left(\begin{bmatrix}x\cdot\nabla u\\x\cdot\nabla v\end{bmatrix}\right) + \theta_{II}^{(u,v)}(w). \tag{58}$$

From equation (58) and Lagnese [26], taking into account the coefficient D and taking only the real parts, we obtain

$$\mathrm{Re}\,\mathrm{a}_{II}(w, x\cdot\nabla w) = \mathrm{a}_{II}(w,w) + \mathscr{T}_6 + \mathscr{T}_7. \tag{59}$$

For the last three terms on the RHS of equation (58), we have

$$S \equiv \mathrm{a}_I\left(\begin{bmatrix}u\\v\end{bmatrix},\begin{bmatrix}x\cdot\nabla u\\x\cdot\nabla v\end{bmatrix}\right) + \theta_I^w\left(\begin{bmatrix}x\cdot\nabla u\\x\cdot\nabla v\end{bmatrix}\right) + \theta_{II}^{(u,v)}(x\cdot\nabla w)$$

$$= K\int_\Omega\left\{u_{x_1}(x\cdot\nabla\bar{u})_{x_1} + \left(v_{x_2}+\frac{1}{R}w\right)\left[(x\cdot\nabla\bar{v})_{x_2} + \frac{1}{R}(x\cdot\nabla\bar{w})\right]\right.$$

$$+\nu\left[u_{x_1}\left((x\cdot\nabla\bar{v})_{x_2}+\frac{1}{R}(x\cdot\nabla\bar{w})\right) + \left(v_{x_2}+\frac{1}{R}w\right)(x\cdot\nabla\bar{u})_{x_1}\right]$$

$$\left.+\frac{1-\nu}{2}(u_{x_2}+v_{x_1})\left((x\cdot\nabla\bar{u})_{x_2} + (x\cdot\nabla\bar{v})_{x_1}\right)\right\}\mathrm{d}x - \frac{K}{R^2}\int_\Omega |w|^2\,\mathrm{d}x.$$

Integrating by parts, using equation (57) and $w|_{\Gamma_0}=0$, we obtain

$$\mathrm{Re}\,S = \mathscr{T}_8 + \mathscr{T}_9 + \mathrm{Re}\,\theta_{II}^{(u,v)}(w). \tag{60}$$

Finally, from equations (59) and (60),

$$\mathrm{Re}\,[\mathrm{a}_{II}(w,w) + \theta_{II}^{(u,v)}(w)] = \mathscr{T}_0,$$

and so every term on the RHS of equation (55) is accounted for.

Now, we are in a position to prove the first main theorem in this paper.

Main Theorem 1 (uniform exponential decay of energy of Donnell's shallow shell). Assume that

(i) F in equation (21) is strictly positive definite, i.e., $F \geqslant \tilde{\beta} I_2$ on Γ_1 for some $\tilde{\beta} > 0$;

$$ \tag{61}$$

(ii) $x \cdot n \leqslant 0$ on Γ_0, (62)

(iii) $x \cdot n \geqslant \gamma > 0$ on Γ_1 for some γ. (63)

Then there exist two positive constants C and μ, independent of the initial state $(w_0, w_1) \in \mathscr{H}$, such that

$$E(t) \leqslant C\,e^{-\mu t} E(0), \qquad t \geqslant 0. \tag{64}$$

Furthermore, if

$$F = \begin{bmatrix} d_1(\cdot) & 0 \\ 0 & 0 \end{bmatrix}, \qquad d_1(x) \geqslant \tilde{\beta} > 0 \quad \text{on } \Gamma_1, \tag{65}$$

then (iii) above can be weakened to $x \cdot n \geqslant 0$, yet with equation (64) remaining valid.

Proof. By Theorem 4.1, we need only prove that there is some $c > 0$ such that

$$\left\| (\mathrm{i}\omega I - \mathscr{A}) \begin{bmatrix} w \\ z \end{bmatrix} \right\|_{\mathscr{H}} \geqslant c \left\| \begin{bmatrix} w \\ z \end{bmatrix} \right\|_{\mathscr{H}}, \qquad \forall \omega \in \mathbb{R}, \qquad \begin{bmatrix} w \\ z \end{bmatrix} \in D(\mathscr{A}).$$

Assume, on the contrary, that the above fails. Then there exist sequences $\omega_p \in \mathbb{R}$, $(w_p, z_p) \in D(\mathscr{A})$ such that

$$|\omega_p| \geqslant \delta > 0 \qquad (\text{because } 0 \in \rho(\mathscr{A})) \tag{66}$$

and

$$\left\| \begin{bmatrix} w_p \\ z_p \end{bmatrix} \right\|_{\mathscr{H}} = 1, \qquad \text{i.e., } \langle\!\langle w_p, w_p \rangle\!\rangle + \langle z_p, z_p \rangle_H = 1, \tag{67}$$

but

$$(\mathrm{i}\omega_p I - \mathscr{A}) \begin{bmatrix} w_p \\ z_p \end{bmatrix} = \begin{bmatrix} f_p \\ g_p \end{bmatrix} \rightarrow 0 \text{ strongly in } \mathscr{H}, \text{ as } p \rightarrow \infty. \tag{68}$$

Equation (68) gives

$$\begin{cases} \mathrm{i}\omega_p w_p - z_p = f_p \rightarrow 0 \text{ strongly in } V, \\ \mathrm{i}\omega_p z_p + (A w_p + B z_p) = g_p \rightarrow 0 \text{ strongly in } H, \end{cases} \quad \text{as } p \rightarrow \infty. \tag{69}$$

We have the regularity $w_p \in H^4(\Omega) \cap H^2_{\Gamma_0}$ and $z_p \in H^2_{\Gamma_0}$. We want to show that equation (69) leads to a contradiction.

We have, from equations (61) or (65),

$$\tilde{\beta}\int_{\Gamma_1}|z_p|^2\,\mathrm{d}\sigma \leqslant \int_{\Gamma_1}\left[z_p, \frac{\partial}{\partial n}z_p\right]F\begin{bmatrix}\bar{z}_p\\ \dfrac{\partial}{\partial n}\bar{z}_p\end{bmatrix}\mathrm{d}\sigma = \langle Bz_p, z_p\rangle_{V^*\times V}$$

$$= \mathrm{Re}\left\langle -\mathscr{A}\begin{bmatrix}w_p\\ z_p\end{bmatrix}, \begin{bmatrix}w_p\\ z_p\end{bmatrix}\right\rangle_{\mathscr{H}}$$

$$= \mathrm{Re}\left\langle (\mathrm{i}\omega_p I - \mathscr{A})\begin{bmatrix}w_p\\ z_p\end{bmatrix}, \begin{bmatrix}w_p\\ z_p\end{bmatrix}\right\rangle_{\mathscr{H}}$$

$$= \mathrm{Re}\left\langle \begin{bmatrix}f_p\\ g_p\end{bmatrix}, \begin{bmatrix}w_p\\ z_p\end{bmatrix}\right\rangle_{\mathscr{H}} = o(1), \tag{70}$$

where $o(1)$ is the little o notation, meaning that the sequence tends to zero. Therefore, if equation (61) holds, we have

$$\|z_p\|_{L^2(\Gamma_1)} = o(1), \quad \left\|\frac{\partial}{\partial n}z_p\right\|_{L^2(\Gamma_1)} = o(1), \quad \|Bz_p\|_{V^*} = o(1), \qquad \text{as } p\to\infty. \tag{71}$$

Let $(u_p, v_p) = \mathscr{L}w_p$ and write $\xi_p = (u_p, v_p, w_p)$. Then equation (69) is equivalent to

$$\begin{cases} z_p = \mathrm{i}\omega_p w_p - f_p, \\ \mathbb{A}\xi_p - \begin{bmatrix}0\\0\\ \omega_p^2 w_p\end{bmatrix} = \begin{bmatrix}0\\0\\ g_p + \mathrm{i}\omega_p f_p\end{bmatrix} & \text{on } \Omega, \\ w_p = \dfrac{\partial}{\partial n}w_p = z_p = 0 & \text{on } \Gamma_0, \\ \begin{cases}\mathscr{B}_1\xi_p = \mathscr{B}_2\xi_p = 0 \\ \begin{bmatrix}B_1 w_p\\ -B_2 w_p\end{bmatrix} = F\begin{bmatrix}z_p\\ \dfrac{\partial}{\partial n}z_p\end{bmatrix} \end{cases} & \text{on } \Gamma_1. \end{cases} \tag{72}$$

Form the $V^*\times V$ pairing-product of equation $(69)_2$ with $\bar{w}_p$ by substituting z_p from equation $(69)_1$; we obtain

$$-\omega_p^2\|w_p\|_H^2 + \langle\!\langle w_p, w_p\rangle\!\rangle = \mathrm{i}\omega_p\langle f_p, w_p\rangle_H - \langle Bz_p, w_p\rangle_{V^*\times V} - \langle g_p, w_p\rangle_H. \tag{73}$$

By equations (43), (67) and (71), we have

$$\langle Bz_p, w_p\rangle_{V^*\times V} \equiv o(1). \tag{74}$$

But $\|g_p\|_H = o(1)$ from equation $(69)_2$, so we have

$$\langle g_p, w_p\rangle_H = o(1). \tag{75}$$

From equations $(69)_1$, (67) and the fact that $\| f_p \|_V = o(1)$, we obtain

$$\mathrm{i}\omega_p \langle f_p, w_p \rangle_H = \langle f_p, -\mathrm{i}\omega_p w_p \rangle_H = \langle f_p, -z_p - f_p \rangle_H = o(1). \tag{76}$$

Using equations (74)–(76) in equation (73), we obtain

$$-\omega_p^2 \| w_p \|_H^2 + \langle\!\langle w_p, w_p \rangle\!\rangle = o(1), \qquad \text{as } p \to \infty. \tag{77}$$

Also, from equation $(69)_1$, we obtain

$$\begin{aligned} \omega_p^2 \int_{\Gamma_1} |w_p|^2 \, \mathrm{d}\sigma &= \int_{\Gamma_1} |z_p + f_p|^2 \, \mathrm{d}\sigma \\ &= \int_{\Gamma_1} |z_p|^2 \, \mathrm{d}\sigma + 2 \operatorname{Re} \int_{\Gamma_1} z_p \overline{f_p} \, \mathrm{d}\sigma + \int_{\Gamma_1} |f_p|^2 \, \mathrm{d}\sigma \\ &= o(1), \end{aligned} \tag{78}$$

by equation (70), and the fact that $\| f_p \|_V \to 0$ and the Trace Theorem.

We form the inner product of equation $(72)_2$ with $(x \cdot \nabla \bar{u}_p, x \cdot \nabla \bar{v}_p, x \cdot \nabla \bar{w}_p)$ and integrate by parts. We first obtain by (78),

$$\int_\Omega (-\omega_p^2 w_p)(x \cdot \nabla \bar{w}_p) \, \mathrm{d}x = \int_\Omega \omega_p^2 |w_p|^2 \, \mathrm{d}x - \omega_p^2 \int_{\Gamma_1} (x \cdot n) |w_p|^2 \, \mathrm{d}\sigma. \tag{79}$$

Second, by Lemma 4.2, we have

$$\begin{aligned} \operatorname{Re} \int_\Omega \mathscr{A} \xi_p \cdot \begin{bmatrix} x \cdot \nabla \bar{u}_p \\ x \cdot \nabla \bar{v}_p \\ x \cdot \nabla \bar{w}_p \end{bmatrix} \mathrm{d}x &= \langle\!\langle w_p, w_p \rangle\!\rangle + \tfrac{1}{2} \mathscr{T}_1(p) + \sum_{j=2}^{3} \mathscr{T}_j(p) + \tfrac{1}{2} \mathscr{T}_4(p) + \mathscr{T}_6(p) \\ &\quad + \mathscr{T}_7(p) + \mathscr{T}_9(p), \end{aligned} \tag{80}$$

where each $\mathscr{T}_j(p)$ is the same as $\mathscr{T}_j$ in equation (55), except that we have substituted (u_p, v_p, w_p) for (u, v, w) in $\mathscr{T}_j$, and where we have simplified equation (55) using $\mathscr{T}_5(p) = 0$, because of the boundary conditions in equation $(72)_4$, and $\mathscr{T}_6(p) = -\frac{1}{2} \mathscr{T}_1(p)$, $\mathscr{T}_8(p) = -\frac{1}{2} \mathscr{T}_4(p)$, by direct comparisons. Third,

$$\begin{aligned} \langle g_p + \mathrm{i}\omega_p f_p, x \cdot \nabla w_p \rangle_H &= o(1) + \mathrm{i}\omega_p \int_\Omega f_p (x \cdot \nabla \bar{w}_p) \, \mathrm{d}x \qquad \text{(by equation (75))} \\ &= o(1) + \mathrm{i}\omega_p \int_{\partial\Omega} (x \cdot n) f_p \bar{w}_p \, \mathrm{d}\sigma - \int_\Omega (2f_p + x \cdot \nabla f_p)(\mathrm{i}\omega_p \bar{w}_p) \, \mathrm{d}x \\ &= o(1), \qquad \text{by equations } (69)_1, (77) \text{ and } (78). \end{aligned} \tag{81}$$

Combining equations (79), (80) and (81), we obtain

$$o(1) = \langle\langle w_p, w_p \rangle\rangle + \tfrac{1}{2}\mathscr{T}_1(p) + \sum_{j=2}^{3} \mathscr{T}_j(p) + \tfrac{1}{2}\mathscr{T}_4(p) + \mathscr{T}_7(p) + \mathscr{T}_9(p) + \omega_p^2 \int_\Omega |w_p|^2 \, \mathrm{d}x$$

$$\geqslant \langle\langle w_p, w_p \rangle\rangle + \omega_p^2 \int_\Omega |w_p|^2 \, \mathrm{d}x + \mathscr{T}_2(p) + \mathscr{T}_3(p) + \mathscr{T}_7(p), \tag{82}$$

because $\mathscr{T}_1(p)$, $\mathscr{T}_4(p)$, $\mathscr{T}_9(p) \geqslant 0$ by equations (62) and (63). Under the assumption of equation (60), there exists a constant $C > 0$ such that

$$|\mathscr{T}_2(p) + \mathscr{T}_3(p)| \leqslant C\epsilon^{-1}\left[\int_{\Gamma_1} (|B_1 w_p|^2 + |B_2 w_p|^2] \, \mathrm{d}\sigma\right] + \varepsilon \mathscr{T}_7(p), \quad \text{for any small } \varepsilon > 0. \tag{83}$$

From equation (83), using equations (66) and $(67)_5$, we obtain

$$|\mathscr{T}_2(p) + \mathscr{T}_3(p)| \leqslant \varepsilon \mathscr{T}_7(p) + o(1). \tag{84}$$

Using equation (84) in equation (82), we obtain

$$\text{LHS of equation (82)} = o(1) \geqslant \langle\langle w_p, w_p \rangle\rangle + \omega_p^2 \int_\Omega |w_p|^2 \, \mathrm{d}x + (1-\varepsilon)\mathscr{T}^7(p)$$

$$\geqslant \langle\langle w_p, w_p \rangle\rangle + \omega_p^2 \int_\Omega |w_p|^2 \, \mathrm{d}x \qquad \text{because } \mathscr{T}_7(p) \geqslant 0)$$

$$= 2\langle\langle w_p, w_p \rangle\rangle + o(1), \qquad \text{by equation (77).} \tag{85}$$

Therefore $\langle\langle w_p, w_p \rangle\rangle = o(1)$, contradicting equations (67), $(72)_1$ and (77).

If, instead of equation (61), we have equation (65), then $\mathscr{T}_3(p) = 0$, and $\mathscr{T}_7(p) \geqslant 0$ in equation (82) (because $x \cdot n \geqslant 0$), and there exists a small $\varepsilon > 0$ such that $\mathscr{T}_2(p) \leqslant \varepsilon \langle\langle w_p, w_p \rangle\rangle$, for all p sufficiently large. Again equation (85) holds, and we have a contradiction.

The proof is complete.

We can further utilize an idea from Komornik and Zuazua [27] to weaken the geometrical condition (59), to obtain the following:

Corollary 4.3. Assume that equations (57) and (58) hold, and that the boundary condition $(20)_3$ is replaced by

$$\begin{bmatrix} B_1 w \\ -B_2 w \end{bmatrix} = (x \cdot n) F \begin{bmatrix} w_t \\ \dfrac{\partial w_t}{\partial n} \end{bmatrix}, \qquad \forall t < 0, \quad \text{on } \Gamma_1,$$

Then equation (59) can be weakened to $x \cdot n \geqslant 0$ on Γ_1, and the uniform exponential decay result (60) of Main Theorem 1 remains valid.

Proof. It is a straightforward matter to check that with the above adjustment of assumptions, all the arguments in the proof of Main Theorem 1 go through. We omit the details.

5. BOUNDARY STABILIZATION OF DONNELL'S SHELLS ON DOMAINS WITH CORNERS

When the Kirchhoff thin plate equation or Donnell's thin circular cylindrical shell system are posed on domains with corners, the corners of such domains may contribute some static strain energy which does not excite vibrations [15]. Therefore our Main Theorem 1 in section 4 is no longer valid. In this section, we properly adapt the arguments so that the corner effects are accounted for and a new uniform exponential decay theorem can be established, under a provisional assumption of *sufficient regularity*.

A corner is a non-smooth point of $\partial\Omega$ where the tangents at that point on the boundary curves from both sides exist and form a non-zero or non-cusp angle. We formulate the following condition.

[DC]: We say that a bounded connected domain $\Omega \subseteq \mathbb{R}^2$ satisfies the [DC] condition if $\partial\Omega$ is C^2-continuous everywhere except at corner points $\{P_j \,|\, j = 1, 2, \ldots, l\}$. The boundary $\partial\Omega$ is the union of two non-empty closed connected subsets Γ_0 and Γ_1, where Γ_0 and Γ_1 are either disjoint, or share two common end points which are corner points.

The basic difference between domains without and domains with corners can be observed in the following lemma.

Lemma 5.1. [15] (integration by parts formula for Donnell's shell on domains with corners). Let Ω satisfy the [DC] condition. Let $\partial\Omega$ be parametrized in a counterclockwise sense. Then, for sufficiently smooth functions u_i, v_i and w_i, $i = 1, 2$, defined on Ω and for the strain energy bilinear form $\mathrm{a}(\cdot, \cdot)$ in equation (12), we have

$$\mathrm{a}\left(\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix}\right) = \text{RHS of (13)} + \sum_{j=1}^{l} [M_T(w_1)(P_j)]\bar{w}_2(P_j), \tag{86}$$

where, for a function w, the twisting moment is

$$M_T(w) = D(1-\nu)\,[n_1 n_2 (w_{x_1 x_1} - w_{x_2 x_2}) - (n_1^2 - n_2^2) w_{x_1 x_2}],$$

$$[M_T(w)(P_j] = \text{the jump of } M_T(w) \text{ across}$$

$$P_j \text{ in the direction of increasing arc length}$$

$$= M_T(w)(P_j^+) - M_T(w)(P_j^-).$$

Note that the extra terms newly appearing on the RHS of equation (86) denote the work done by the l corner forces $[M_T(w_1)](P_j)$, $j = 1, 2, \ldots, l$, acting through the l corner displacements $w_2(P_j)$ [13–15].

Next, we derive dissipative boundary conditions. We have

$$\frac{d}{dt}E(t) = \frac{d}{dt}\left\{\int_\Omega m w_t^2\, dx + a\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}, \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right)\right\}$$

$$= \text{RHS of (17)} + \sum_{j=1}^{l} [M_T(w)(P_j)]\bar{w}_t(P_j). \tag{87}$$

Therefore, if we prescribe that

$$\begin{bmatrix} w \\ \dfrac{\partial w}{\partial n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{on } \Gamma_0; \tag{88}$$

$$\begin{bmatrix} B_1 w \\ -B_2 w \end{bmatrix} = F \begin{bmatrix} w_t \\ \dfrac{\partial}{\partial n} w_t \end{bmatrix}, \quad F \text{ is the same as in equation (21)}, \quad \text{on } \Gamma_1 \backslash \{P_1, \ldots, P_l\}; \tag{89}$$

$$[M_T(w)(P_i)] = -\gamma_i w_t(P_i), \qquad \text{if } P_i \in \Gamma_1, \quad \gamma_i \geqslant 0, \quad \forall t > 0, \tag{90}$$

then equation (87) leads to

$$\frac{d}{dt}E(t) = -\int_{\Gamma_1}\left[\bar{w}_t, \frac{\partial}{\partial n}\bar{w}_t\right] F \begin{bmatrix} w_t \\ \dfrac{\partial}{\partial n} w_t \end{bmatrix} d\sigma - \sum_{P_i \in \Gamma_1 \backslash \Gamma_0} \gamma_i |w_t^2|(P_i) \leqslant 0,$$

and so energy is decreasing.

It is important to note tht in this section, $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ may share corner points together; c.f., Figure 4.

Conditions in equation (90) require that the pointwise limits of the twisting moments $M_T(w)(P_i^-)$, $M_T(w)(P_i^+)$, as well as the pointwise values $w_t(P_i)$, exist for $i = 1, 2, \ldots, l$.

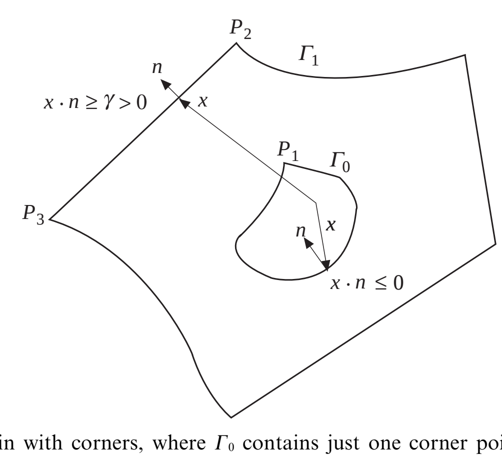


Figure 4. A domain with corners, where Γ_0 contains just one corner point, and $\Gamma_0 \cap \Gamma_1 = \emptyset$.

If $w \in H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)$ and $w_t \in H^2_{\Gamma_0}(\Omega)$ are true, then Sobolev's imbedding theorem will give

$$\begin{cases} w \in C^{2,\alpha_1}(\partial\Omega) & \text{for any } \alpha_1 : 0 < \alpha_1 < 1, \\ w_t \in C^{0,\alpha_2}(\partial\Omega)' & \text{for any } \alpha_2 : 0 < \alpha_2 < 1, \end{cases}$$

and therefore there would be no problem in equation (90). However, because $\partial\Omega$ here contains corners, the classical regularity results for solutions of elliptic boundary value problems may no longer be valid in general; the corner conditions in equation (90) may not be well-defined in the pointwise sense; see Remark 5.3. This difficulty may be overcome by again using the $V \hookrightarrow H \hookrightarrow V^*$ formalism. For $w_1, w_2 \in V$, define a sesquilinear form

$$\mathbb{b}_c(w_1, w_2) = \sum_{P_i \in \Gamma_1} \gamma_i \, w_1(P_i) \bar{w}_2(P_i).$$

(The subscript "c" here and later refers to "corner".) Then $\mathbb{b}_c$ induces an operator B_c:

$$B_c \colon V \equiv H^2_{\Gamma_0}(\Omega) \to V^*, \qquad \mathbb{b}_c(v_1, v_2) = \langle B_c v_1, v_2 \rangle_{V^* \times V}.$$

Obviously B_c is a symmetric positive semidefinite operator satisfying

$$\langle B_c v, v \rangle_{V^* \times V} \geqslant 0, \qquad \forall v \in V.$$

Let A and B be defined, respectively, as in equations (40) and (44). Then we have the following theorem.

Theorem 5.2. Let Ω satisfy [DC]. Define an operator $\mathscr{A}_c$ on $\mathscr{H} = V \times H$ $(H = L^2(\Omega))$,

$$\mathscr{A}_c = \begin{bmatrix} 0 & I \\ -A & -(B + B_c) \end{bmatrix} : D(\mathscr{A}_c) \to \mathscr{H},$$

$$D(\mathscr{A}_c) = \left\{ \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \middle| w_0, w_1 \in V, \, -[Aw_0 + (Bw_1 + B_c w_1)] \in H \right\}.$$

Then $\mathscr{A}_c$ is the infinitesimal generator of a C_0-semigroup of contractions in $\mathscr{H}$. Furthermore, for any initial condition $(w_0, w_1) \in D(\mathscr{A}_c)$, the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} w \\ \dot{w} \end{bmatrix} = \mathscr{A}_c \begin{bmatrix} w \\ \dot{w} \end{bmatrix}, \qquad t > 0$$

$$\begin{bmatrix} w(\cdot\,, 0) \\ \dot{w}(\cdot\,, 0) \end{bmatrix} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \tag{91}$$

satisfies

$$w \in C^1([0, \infty); V) \cap C^2([0, \infty); H),$$

$$Aw + (B\dot{w} + B_c \dot{w}) \in C([0, \infty); H),$$

$$\ddot{w} + Aw + (B\dot{w} + B_c \dot{w}) = 0, \qquad t > 0.$$

Proof. The arguments are the same as in reference [1] and in section 3.

Remark 5.3. Let us comment on some important regularity issues here. Assume that [DC] holds, and $(f, g) \in \mathscr{H}$. For given $\lambda \in \mathbb{C}$, consider the resolvent equation

$$\begin{gathered} \text{Find } (w_0, w_1) \in D(\mathscr{A}_c) \text{ s.t.} \\ (\mathscr{A}_c - \lambda I_2)\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \in \mathscr{H} \end{gathered}. \tag{92}$$

If Ω does not have corners, then for $\mathscr{A}$ as defined in section 4 the solution of

$$\begin{gathered} \text{Find } (w_0, w_1) \in D(\mathscr{A}), \\ (\mathscr{A} - \lambda I_2)\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \in \mathscr{H} \end{gathered}.$$

has regularity $w_0 \in H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)$, $w_1 \in H^2_{\Gamma_0}(\Omega)$. This is a consequence of the theory of classical elliptic boundary value problems. However, for equation (92), by straightforward variational analysis, it is easy to show that if equation (92) has a unique solution (w_0, w_1), and if w_0 is sufficiently regular, then w_0 and $(u, v) = \mathscr{L} w_0$ satisfy

$$\begin{gathered} u_{x_1x_1} + \frac{1+\nu}{2} v_{x_1x_2} + \frac{1-\nu}{2} u_{x_2x_2} + \frac{\nu}{R} w_{0x_1} = 0, \qquad \text{on } \Omega, \\ \frac{1+\nu}{2} u_{x_1x_2} + \frac{1-\nu}{2} v_{x_1x_1} + v_{x_2x_2} + \frac{1}{R} w_{0x_2} = 0, \qquad \text{on } \Omega, \\ D\left[-\Delta^2 w_0 - \frac{12}{Rh^2}\left(\frac{1}{R} w_0 + \nu u_{x_1} + v_{x_2} \right)\right] - \lambda^2 w_0 = \lambda f + g, \qquad \text{on } \Omega, \\ u = 0, \quad v = 0, \quad \text{on } \Gamma_0, \\ \mathscr{B}_1(u, v, w_0) = 0, \quad \mathscr{B}_2(u, v, w_0) = 0, \quad \text{on } \Gamma_1, \\ \begin{bmatrix} B_1 w_0 \\ -B_2 w_0 \end{bmatrix} - \lambda F \begin{bmatrix} w_0 \\ \dfrac{\partial}{\partial n} w_0 \end{bmatrix} = F \begin{bmatrix} f \\ \dfrac{\partial}{\partial n} f \end{bmatrix} \qquad \text{on } \Gamma_1, \end{gathered} \tag{93}$$

$$[M_t(w_0)(P_i)] + \lambda \gamma_i w_0(P_i) = -\gamma_i f(P_i), \qquad P_i \in \Gamma_1, \tag{94}$$

$$u, v \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega), \qquad w_1 = f + \lambda w_0 \in H^2_{\Gamma_0}(\Omega).$$

In order for the pointwise limits of w_0 in equation (94) to exist, a sufficient condition is that $w_0 \in H^{3+\varepsilon}(\Omega)$ for some ε: $0 < \varepsilon \leqslant 1$; i.e., the presence of corners causes the loss of regularity at most of Sobolev space order $1 - \varepsilon$. But we also note that for the energy identity in Lemma 5.5 to work, we need a higher regularity: $w \in H^{7/2+\varepsilon}(\Omega)$, for some ε: $0 < \varepsilon \leqslant 1/2$.

Solutions of elliptic boundary value problems on (curvilinear) polygonal or Lipschitz domains are known to lose regularity. For polygonal domains, the most effective method for studying regularity of solutions was developed by Kondratiev [28]; see also Kondratiev and Oleinik [29]. More recent results are available in Grisvard [30, 31]. For *second order* elliptic problems, we have learned from these references that on *convex polygonal* domains, solutions do not lose regularity. But on non-convex polygonal domains, solutions lose regularity only for a *discrete set of angular values*.

Although references [30, 31] did mention some results for fourth order problems, the boundary conditions therein are of the lower order and thus not directly applicable to our case of interest here.

The closest relevant reference to the question of regularity in this section may be found in Blum and Rannacher [32], where they studied the effects of corners on the possible loss of regularity of (perturbed) biharmonic boundary value problems subject to boundary conditions involving B_1 and B_2. Their results seem to suggest a situation analogous to the second order elliptic case; namely, that on polygonal domains, except for a discrete set of angular values (see reference [32, Theorem 2, p. 563]), the solutions do not lose any regularity, and consequently these results offer strong support that there do exist many domains on which equations (92) or (93) and therefore equations (94) are indeed satisfied in the pointwise sense. S. Nicaise [33, 34] further extended the work of reference [32] to interface problems. If the boundary conditions of the polygonal domain are of lower order (such as the clamped case) treated in Nicaise [33, (5.1)–(5.7), p. 348], for example, then the regularity is known [33, (6.17), p. 357]. However, if part of the boundary conditions are of higher order (such as the free case in reference [33, (6.21), p. 359]), then as Nicaise [33, p. 359] stated, the sharp regularity is still unclear.

We also note that the variational forms for the biharmonic problems on polygonal domains as give by Nicaise [34, (5.11) *et al.*, p. 178] are somewhat restrictive. The most general form can be found in Schmidt [35]. Based on his work, Chen, Coleman and Ding [15] have shown that "extraneous" weak solutions of the biharmonic boundary value problem occur if additional pointwise constraints are not imposed at the vertices of the edges where the boundary conditions are free. Those extraneous solutions are not necessarily singular, as reference [15, Example 4.2] has shown. Such extraneous solutions will be eliminated by our feedback conditions at the vertices (90).

The occurrence of *singular* solutions for boundary value problems on non-smooth domains poses considerable technical difficulty. From the control theory viewpoint, we ask whether one can design a feedback law that simultaneously eliminates singular solutions and achieves exponential stabilization? Our work in this section has not yet touched upon this topic. It constitutes a major challenge for the distributed parameter control theorist.

Remark 5.4. Let us take a closer look at boundary conditions (88)–(90). We first show that

$$[M_T(w)(P_j)] = 0, \qquad \text{if } P_j \in \Gamma_0.$$

On Γ_0, we have $w = \partial w/\partial n = 0$. Therefore, on each arc $\widehat{P_j P_{j+1}} \subset \Gamma_0$, where $P_j,\ P_{j+1} \in \Gamma_0$, we have

$$w_{x_1x_1} = n_1^2 \frac{\partial^2 w}{\partial n^2}, \qquad w_{x_1x_2} = n_1 n_2 \frac{\partial^2 w}{\partial n^2}, \qquad w_{x_2x_2} = n_2^2 \frac{\partial^2 w}{\partial n^2},$$

where $n = (n_1, n_2)$ is the unique outward pointing normal on $\widehat{P_j P_{j+1}}$. We obtain (the one-sided limit)

$$\begin{aligned} M_T(w)(P_i^+) &= D(1-\nu)\cdot[n_1 n_2 (w_{x_1x_1} - w_{x_2x_2}) - (n_1^2 - n_2^2) w_{x_1x_2}]|_{x=P_i^+} \\ &= D(1-\nu)\cdot\left[n_1 n_2 (n_1^2 - n_2^2)\left(\frac{\partial^2 w}{\partial n^2} - \frac{\partial^2 w}{\partial n^2}\right)\right]\Bigg|_{x=P_i^+} \\ &= 0. \end{aligned}$$

Similarly,

$$M_T(w)(P_{i+1}^-) = 0.$$

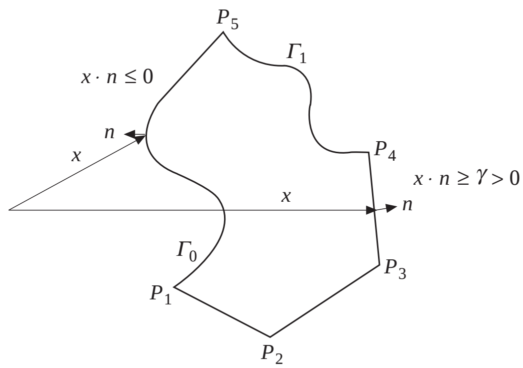


Figure 5. A domain with corners, where Γ_0 and Γ_1 shares two end points which are corner points.

If Γ_0 contains only one corner, say P_1, and if Γ_0 is a closed curve (cf. Figure 4), then instead of considering $\widehat{P_j P_{j+1}}$, we may consider $\widehat{P_1^+ P_1^-}$, the entire curve of Γ_0 starting from P_1^+, ending at P_1^-, and being traced counterclockwise. Then the above argument, with slight modification, still gives

$$M_T(w)(P_1^+) = 0, \qquad M_T(w)(P_1^-) = 0.$$

If Γ_0 and Γ_1 share two end points which are corner points (cf. Figure 5), then because $w = \partial w/\partial n = 0$ on Γ_0, using exactly the same arguments as above, we again obtain $[M_T(w)(P_i)] = 0$ if $P_i \in \Gamma_0 \setminus \Gamma_1$, and if $P_i \in \Gamma_0 \cap \Gamma_1$, we obtain

$$\text{either} \qquad M_T(w)(P_i^+) = 0, \qquad \text{or} \qquad M_T(w)(P_i^-) = 0, \tag{95}$$

depending on whether $P_i^+ \in \Gamma_0$ or $P_i^- \in \Gamma_0$, respectively. Let us denote

$$\{P_{i_1}, P_{i_2}\} = \Gamma_0 \cap \Gamma_1, \qquad \text{where } P_{i_1}^+, P_{i_2}^- \in \Gamma_0, \qquad i_1, i_2 \in \{1, 2, \ldots, l\}.$$

Then, from equation (90), we obtain

$$\begin{aligned}
[M_T(w)(P_{i_1})] &= M_T(w)(P_{i_1}^+) - M_T(w)(P_{i_1}^-) \\
&= -M_T(w)(P_{i_1}^-) \\
&= -\gamma_{i_1} w_t(P_{i_1}) \qquad \text{(by equation (95))} \\
&= 0,
\end{aligned}$$

because w_t is continuous on $\partial\Omega$ and $0 = w_t(P_{i_1}^+) = w_t(P_i^-)$. Similarly,

$$[M_T(w)(P_{i_2})] = 0.$$

Lemma 5.5. (energy identity for domains with corners). Let Ω satisfy [DC]. Let $w \in H^{7/2+\varepsilon}(\Omega)$ and let $u, v \in H^{3/2+\varepsilon}(\Omega)$ for some small $\varepsilon > 0$. Assume that $w = \partial w/\partial n = 0$ and $u = v = 0$ on Γ_0. Then

$$\operatorname{Re} \int_\Omega \mathbb{A} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \cdot \begin{bmatrix} x \cdot \nabla \bar{u} \\ x \cdot \nabla \bar{v} \\ x \cdot \nabla \bar{w} \end{bmatrix} \mathrm{d}x = \sum_{j=0}^{10} \mathscr{T}_j,$$

where $\mathscr{T}_j$, $j = 0, 1, 2, \ldots, 9$, ae the same as in equation (55), and

$$\mathscr{T}_{10} \equiv -\mathrm{Re} \sum_{P_j \in \Gamma_1 \backslash \Gamma_0} [M_T(w)(P_j)] \cdot (x \cdot \nabla \bar{w})|_{x=P_j}.$$

Proof. Using Lemmas 3.1 and 5.1 first, we obtain

$$\int_\Omega \mathbb{A}\begin{bmatrix} u \\ v \\ w \end{bmatrix} \cdot \begin{bmatrix} x \cdot \nabla \bar{u} \\ x \cdot \nabla \bar{v} \\ x \cdot \nabla \bar{w} \end{bmatrix} \mathrm{d}x = \mathrm{a}\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix} \cdot \begin{bmatrix} x \cdot \nabla \bar{u} \\ x \cdot \nabla \bar{v} \\ x \cdot \nabla \bar{w} \end{bmatrix}\right)$$

$$- \int_{\partial\Omega} \Big\{ \mathscr{B}_1(u, v, w) \cdot (x \cdot \nabla \bar{u}) + \mathscr{B}_2(u, v, w) \cdot (x \cdot \nabla \bar{v}) \}$$

$$- (B_1 w)(x \cdot \nabla \bar{w}) + (B_2 w) \frac{\partial}{\partial n}(x \cdot \nabla \bar{w}) \Big\} \mathrm{d}\sigma$$

$$- \sum_{P_j \in \Gamma_1 \backslash \Gamma_0} [M_T(w)(P_j)] \cdot (x \cdot \nabla \bar{w})|_{x=P_j}. \tag{96}$$

Under the given regularity assumptions on u, v and w, the treatment of the RHS of equation (96) (except for the last sum, $\mathscr{T}_{10}$), is exactly the same as in the proof of Lemma 4.2. Therefore the proof is complete.

To make it possible to establish an exponential decay theorem using the frequency domain method as in section 5, we need a regularity assumption on the solution of the resolvent equation as follows.

[$\mathscr{R}$] Let Ω satisfy [DC]. Let $(f, g) \in \mathscr{H}$. Let $(w_0, w_1) \in D(\mathscr{A}_c)$ satisfy

$$(\lambda I - \mathscr{A}_c)\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \quad \text{in } \mathscr{H},$$

for some $\lambda \in \mathbb{C}$, and let $(u, v) = \mathscr{L} w_0$. Then $w_0 \in H^{7/2+\varepsilon}(\Omega)$, $u, v \in H^{3/2+\varepsilon}$, for some $\varepsilon > 0$.

With the help of [$\mathscr{R}$], we can now achieve our final result.

Main Theorem 2. (uniform exponential decay of energy of Donnell's shallow shell on domains with corners). Assume [$\mathscr{R}$] as well as conditions (i), (ii) and (iii) in Main Theorem 1. Let (w, w_t) be the solution of equation (91) with initial state $(w_0, w_1) \in \mathscr{H}$. Then there exist two positive constants C and μ, independent of $(w_0, w_1) \in \mathscr{H}$, such that

$$E(t) \leqslant C\,\mathrm{e}^{-\mu t} E(0).$$

Furthermore, if equation (65) holds and if $\gamma_i = 0$ in equation (90) for all γ_i, then assumption (iii) can be weakened to $x \cdot n \geqslant 0$ on Γ_1.

Proof. Assumption [$\mathscr{R}$] takes care of all the required smoothness in energy multiplier manipulations, and we will be able to give a proof in pretty much the same way as that of Main Theorem 1; we need only watch the minor differences. First, note that equation (70) now becomes

$$\overline{\beta}\int_{\Gamma_1}|z_p|^2\,\mathrm{d}\sigma+\sum_{P_j\in\Gamma_1\setminus\Gamma_0}\gamma_j\,|z_p(P_j)|^2$$

$$\leqslant\int_{\Gamma_1}\left[z_p\;\frac{\partial}{\partial n}z_p\right]F\begin{bmatrix}\bar{z}_p\\ \dfrac{\partial}{\partial n}\bar{z}_p\end{bmatrix}\mathrm{d}\sigma+\sum_{P_j\in\Gamma_1\setminus\Gamma_0}\gamma_j\,z_p(P_j)\bar{z}_p(P_j)$$

$$=\langle(B+B_c)z_p,z_p\rangle_{V^*\times V}=\mathrm{Re}\left\langle-\mathscr{A}\begin{bmatrix}w_p\\ z_p\end{bmatrix},\begin{bmatrix}w_p\\ z_p\end{bmatrix}\right\rangle_{\mathscr{H}}$$

$$=\cdots$$

$$=o(1).$$

Therefore, under assumption (61), we have equation (71) further strengthened to

$$\|z_p\|_{L^2(\Gamma_1)}=o(1),\qquad\left\|\frac{\partial}{\partial n}z_p\right\|_{L^2(\Gamma_1)}=o(1),\qquad\|Bz_p\|_{V^*}=o(1),$$

$$|z_p(P_j)|=o(1),\qquad\text{for }\;P_j\in\Gamma_j\setminus\Gamma_0,\quad\text{if }\gamma_j>0.\tag{97}$$

Next, observe that equation (73) becomes

$$-\omega_p^2\,\|w_p\|_H^2+\langle\!\langle w_p,w_p\rangle\!\rangle-\sum_{P_j\in\Gamma_1\setminus\Gamma_0}[M_T(w_p)(P_j)]\bar{w}_p(P_j)$$

$$=\mathrm{i}\omega_p\langle f_p,w_p\rangle_H-\langle Bz_p,w_p\rangle_{V^*\times V}+\langle g_p,w_p\rangle_H.\tag{98}$$

The third term on the LHS of equation (98), by equation (94) with $\lambda=\mathrm{i}\omega$, $f=f_p$ and $w_0=w_p$ therein, is equal to

$$-\sum_{p_j\in\Gamma_1\setminus\Gamma_0}[M_T(w_p)(P_j)]\bar{w}_p(P_j)=\sum_{P_j\in\Gamma_1\setminus\Gamma_0}\mathrm{i}\omega_p\gamma_j\,|w_p(P_j)|^2+\sum_{P_j\in\Gamma_1\setminus\Gamma_0}\gamma_j f(P_j)\bar{w}_p(P_j).\tag{99}$$

We rewrite equation (98) using equation (99) to obtain

$$-\omega_p^2\,\|w_p\|_H^2+\langle\!\langle w_p,w_p\rangle\!\rangle+\mathrm{i}\sum_{P_j\in\Gamma_1\setminus\Gamma_0}\gamma_j\,\omega_j\,|w_p(P_j)|^2$$

$$=-\sum_{P_j\in\Gamma_1\setminus\Gamma_0}\gamma_j f_p(P_j)\bar{w}_p(P_j)+\text{RHS of equation (73).}\tag{100}$$

We can prove that the RHS of equation (98) is $o(1)$ by using the same argument as in equation (73)–(77) and the Trace Theorem. Therefore, by taking the real and imaginary parts of the LHS of equation (100), we obtain

$$-\omega_p^2\,\|w_p\|_H^2+\langle\!\langle w_p,w_p\rangle\!\rangle=o(1),\;\sum_{P_j\in\Gamma_1\setminus\Gamma_0}\gamma_p\,\omega_p\,|w_p(P_j)|^2=o(1).\tag{101}$$

Third, note by Lemma 5.5 that equation (80) is now modified to

$$\operatorname{Re}\int_{\Omega} \mathscr{A}\xi_p \cdot \begin{bmatrix} x \cdot \nabla \bar{u}_p \\ x \cdot \nabla \bar{v}_p \\ x \cdot \nabla \bar{w}_p \end{bmatrix} \mathrm{d}x = \text{RHS of (80)} + \mathscr{T}_{10}(p). \tag{102}$$

But

$$\begin{aligned} \mathscr{T}_{10}(p) &= -\operatorname{Re} \sum_{P_j \in \Gamma_1 \backslash \Gamma_0} [M_T(w_p)(P_j)] \cdot (x \cdot \nabla \bar{w}_p)|_{x=P_j} \\ &= \operatorname{Re}\Bigg[\mathrm{i}\omega_p \sum_{P_j \in \Gamma_1 \backslash \Gamma_0} \gamma_j w_p(P_j) \cdot (x \cdot \nabla \bar{w}_p)|_{x=P_j} \\ &\quad + \sum_{P_j \in \Gamma_1 \backslash \Gamma_0} \gamma_j f_p(P_j) \cdot (x \cdot \nabla \bar{w}_p)|_{x=P_j} \Bigg]. \end{aligned} \tag{103}$$

Using equation $(69)_1$, we have

$$\sum_{P_j \in \Gamma_1 \backslash \Gamma_0} \omega_p^2 |w_p(P_j)|^2 \leqslant 2 \sum_{P_j \in \Gamma_1 \backslash \Gamma_0} [|z_p(P_j)|^2 + |f_p(P_j)|^2] = o(1),$$

similarly as in equation (78), by the second line of equation (97) and the Trace Theorem. Therefore we can now estimate both sums on the RHS of equation (103) as follows:

$$\begin{aligned} \left| \mathrm{i}\omega_p \sum \gamma_j w_p(P_j) \cdot (x \cdot \nabla \bar{w}_p)_{x=P_j} \right| &\leqslant \frac{1}{2\alpha} \sum \omega_p^2 |w_p(P_j)|^2 + \frac{\alpha}{2} \sum |(x \cdot \nabla w)_{x=P_j}|^2 \\ &\leqslant o(1) + \alpha' \mathscr{T}_7(p) + \alpha'' \langle\!\langle w_p, w_p \rangle\!\rangle, \end{aligned} \tag{104}$$

$$\begin{aligned} \left| \sum \gamma_j f_p(P_j) \cdot (x \cdot \nabla \bar{w}_p)_{x=P_j} \right| &\leqslant \frac{1}{2\alpha} \sum |f_p(P_j)|^2 + \frac{\alpha}{2} \sum |(x \cdot \nabla w)_{x=P_j}|^2 \\ &\leqslant o(1) + \alpha' \mathscr{T}_7(p) + \alpha'' \langle\!\langle w_p, w_p \rangle\!\rangle, \end{aligned} \tag{105}$$

for some sufficiently small positive α, α' and α''. Therefore the RHS of equations (104) and (105) can be absorbed just as in equations (82)–(85).

The rest of the proof follows as in that of Main Theorem 1.

It goes without saying that an analogue of Corollary 4.3 can also be stated.

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